FREE RESOLUTIONS OF ORBIT CLOSURES FOR REPRESENTATIONS WITH FINITELY MANY ORBITS

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by

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ABSTRACT OF DISSERTATION

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Abstract

The irreducible representations of reductive groups with finitely many orbits are parametrized by graded simple Lie algebras. For the exceptional Lie algebras, Kraśkiewicz and Weyman exhibit the Hilbert polynomials and the expected minimal free resolutions of the normalization of the orbit closures. We present an interactive method to construct explicitly these and related resolutions in Macaulay2. The method is then used in the cases of the Lie algebras of type E_6, F_4, G_2, and select cases of type E_7 to confirm the shape of the expected resolutions as well as some geometric properties of the orbit closures.

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Finally, none of this would be possible without the constant love and support of my parents. It is to them that I dedicate my thesis.

To my parents

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Introduction

In [10], Kac classified the irreducible representations of complex reductive groups with finitely many orbits. These are referred to as representations of type I and are parametrized (with few exceptions) by pairs (X_n, x), where X_n is a Dynkin diagram and x is a "distinguished" node of X_n. The choice of a distinguished node induces a grading on the root system of type X_n and the corresponding simple Lie algebra

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

satisfying the following:

- the Cartan subalgebra $\mathfrak{h}$ is contained in $\mathfrak{g}_0$;
- the root space $\mathfrak{g}_\beta$ is contained in $\mathfrak{g}_i$, where i is the coefficient of the simple root α, corresponding the node x, in the expression of β as a linear combination of simple roots;
- the grading is compatible with the Lie bracket of $\mathfrak{g}$.

In particular, $\mathfrak{g}_0$ is a Lie subalgebra of $\mathfrak{g}$.

Let G be the adjoint group of $\mathfrak{g}$. The representation associated to the pair (X_n, x) is given by the vector space $\mathfrak{g}_1$ with the action of the group G_0, a connected reductive subgroup of G with Lie algebra $\mathfrak{g}_0$.

The orbit closures of the representations of type I were classified by Vinberg. In [18], he showed that the orbits are the irreducible components of the intersections of the nilpotent orbits in $\mathfrak{g}$ with the graded component $\mathfrak{g}_1$. Later, in [19], Vinberg gave a combinatorial description of the orbits in terms of some graded subalgebras of $\mathfrak{g}$.

The study of the minimal free resolutions of these orbit closures goes back to Lascoux and his paper on determinantal varieties [13]. Józefiak, Pragacz and Weyman developed the case of determinantal ideals of symmetric and antisymmetric matrices [9]. The more general case of rank varieties for the classical types was studied by Lovett [14]. For the exceptional types, Kraśkiewicz and Weyman [11] calculate the Hilbert functions of the normalization of the orbit closures. They also describe the terms of the expected resolutions of the normalizations as G_0-representations.

The goal of this work is to introduce a framework for constructing the complexes of Kraśkiewicz and Weyman explicitly. This involves computational methods, carried out in the software system Macaulay2 [5], and techniques from representation theory, commutative algebra and algebraic geometry. As a result, we can verify the shape of the free resolutions conjectured by Kraśkiewicz and Weyman in many cases, whenever our computations are feasible; moreover, we obtain the minimal free resolutions of the coordinate rings of the orbit closures and confirm results about Cohen-Macaulay and Gorenstein orbit closures. Finally, we establish containment and singularity of the orbit closures. In particular, we examine the exceptional Lie algebras of type E_6, F_4, G_2, and select cases of type E_7, and present a detailed analysis of orbit closures for those cases.

Here is an outline of this work. In chapter 1, we provide an overview of the results of Kac and Vinberg, as well as the geometric technique used in the work of Kraśkiewicz and Weyman to extract information about the orbit closures. In chapter 2, we describe the computations ran in Macaulay2; in particular, we introduce the *interactive method for syzygies* and the *cone procedure* for non normal orbit closures. In chapter 3, we discuss the *equivariant exactness criterion*, a simple method to establish exactness of our complexes. In chapter 4, we explain how to obtain the defining equations for the orbit closures and verify they generate a radical ideal. Chapter 5 contains an example worked out in detail. Finally, the remaining chapters describe the free resolutions for the orbit closures in the representations associated with gradings on the Lie algebras of type E_6, F_4, G_2, and select cases of type E_7. Appendix A contains a brief description of the equivariant maps used to construct the differentials in our complexes, while appendix B outlines a strategy to recover the equivariant form of a free resolution obtained with computational methods.

This work is accompanied by a collection of Macaulay2 files containing the defining ideals for the orbit closures as well as presentations of related modules. All files are available online at http://www.math.neu.edu/~fgaletto/orbits[1].

[1]The address above is not available anymore. As of 9/3/2013 all files can be found at http://www.mast.queensu.ca/~galetto/orbits

Chapter 1

Preliminaries

In this chapter, we recall some preliminary results of Kac and Vinberg regarding representations of complex reductive groups with finitely many orbits. In particular, our case by case analysis carried out in the later chapters is based upon the classification presented here. We also provide an overview of the geometric technique used by Kraśkiewicz and Weyman to study minimal free resolutions related to the orbit closures in these representations.

1.1 Representations with finitely many orbits

Let $\mathfrak{g}$ be a complex simple Lie algebra. We refer the reader to [7] for an introduction to the theory of Lie algebras. Fixing a Cartan subalgebra $\mathfrak{h}$ determines an irreducible root system Φ and a root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha \right).$$

By choosing a base Δ of Φ, we can associate to Φ a (connected) Dynkin diagram. Recall that the connected Dynkin diagrams are classified by the following (families of) types: A_n, B_n, C_n, D_n, for a positive integer n, (the classical types) and E_6, E_7, E_8, F_4, G_2 (the exceptional types).

We can define a grading on $\mathfrak{g}$ as follows. Let $\alpha_k \in \Phi$ be a simple root. The choice of α_k is equivalent to selecting a node on the Dynkin diagram of type X_n; we call this the *distinguished* node. We refer to the nodes of a diagram by the standard numbering of [1]. For any root $\beta \in \Phi$, we have

$$\beta = \sum_{\alpha_i \in \Delta} c_i \alpha_i,$$

where $c_i \in \mathbb{Z}$. Define the degree of β with respect to the pair (X_n, α_k) to be the integer c_k. If $\mathfrak{g}_i$ is taken to be the direct sum of all root spaces of $\mathfrak{g}$ corresponding to roots of degree i, then we have a graded decomposition:

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

with the following properties:

- $\mathfrak{h} \subseteq \mathfrak{g}_0$;
- $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$;
- $\mathfrak{g}_{-i} \cong \mathfrak{g}_i^*$, where the duality is taken with respect to the Killing form of $\mathfrak{g}$.

The graded component $\mathfrak{g}_0$ in the graded decomposition of $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{g}$. If G is the adjoint group of $\mathfrak{g}$, then G contains a connected reductive subgroup G_0 whose Lie algebra is $\mathfrak{g}_0$. In particular, the group G_0 and the Lie algebra $\mathfrak{g}_0$ have Dynkin diagram given by X_n minus the distinguished node. Kac proved in [10] that the action of G_0 on $\mathfrak{g}$ respects the grading and G_0 acts on each graded component $\mathfrak{g}_i$ with a finite number of orbits. In the same paper, Kac also proved that a connected linear algebraic group acting on an irreducible representation with finitely many orbits must be one of the following cases:

1. $G_0 \circlearrowright \mathfrak{g}_1$, where G_0 and $\mathfrak{g}_1$ arise from a pair (X_n, α_k) as outlined above;
2. $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_3(\mathbb{C}) \times \mathrm{SL}_n(\mathbb{C}) \times \mathbb{C}^\times \circlearrowright \mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^n$, for $n \geqslant 6$;
3. $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{Spin}(7) \circlearrowright \mathbb{C}^2 \otimes V_{\omega_3}$, with V_{ω_3} the third fundamental representation of $\mathrm{SO}_7(\mathbb{C})$.

In this work, we will focus on the representations in the first case, where X_n is an exceptional type.

Example 1.1.1. Let $(X_n, \alpha_k) = (A_{m+n+1}, \alpha_m)$. In this case, $\mathfrak{g} = \mathfrak{sl}_{m+n}(\mathbb{C})$ and the grading is:

$$\mathfrak{sl}_{m+n}(\mathbb{C}) = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1,$$

where:

- $\mathfrak{g}_0 \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{sl}_m(\mathbb{C}) \oplus \mathbb{C}$;
- $\mathfrak{g}_1 \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$.

Here the group G_0 is $\mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C}) \times \mathbb{C}^\times$. We can think of the representation $G_0 \circlearrowright \mathfrak{g}_1$, as the space of $m \times n$ matrices with the action corresponding to a simultaneous change of basis in the domain and the codomain. Two matrices are in the same orbit of the action if and only if they have the same rank. In particular, there are finitely many orbits.

Remark 1.1.2. It is important to stress that the representations we consider arise from a grading on a simple Lie algebra of a given type X_n, however they carry the action of a group whose type is $X_n \backslash \{\alpha_k\}$. In particular, even though we consider the cases parametrized by the exceptional types, the groups acting are hardly ever exceptional.

1.2 Classification of the orbits

Let us fix a representation $G_0 \circlearrowright \mathfrak{g}_1$, arising from a pair (X_n, α_k) as described in the previous section. Let $e \in \mathfrak{g}_1$ be a nilpotent element of $\mathfrak{g}$ and $C(e)$ its conjugacy class. The intersection $C(e) \cap \mathfrak{g}_1$ breaks into a union of irreducible components

$$C_1(e) \cup \ldots \cup C_{n(e)}(e).$$

The sets $C_i(e)$ are G_0-stable. Vinberg proved in [18] that the orbits for the action of G_0 on $\mathfrak{g}_1$ are precisely the $C_i(e)$, for all possible choices of conjugacy classes $C(e)$.

Although Vinberg's result gives a complete description of the G_0-orbits on $\mathfrak{g}_1$, it is not very easy to use in practice. Vinberg also gave another way of classifying the orbits in [19].

This approach consists in classifying the support of nilpotent elements of $\mathfrak{g}$, i.e. certain graded subalgebras of $\mathfrak{g}$. Moreover, this can be turned into a more combinatorial problem which can be solved explicitly for any particular grading on a simple Lie algebra. Kraśkiewicz and Weyman [11] provide a list of all the orbits, together with a representative, in the representations corresponding to gradings on the simple Lie algebras of type E_6, F_4 and G_2 (analogous results for E_7 and E_8 will appear in future work). For ease of access, we provide their results when needed in the later chapters. The reader may consult [11] for a more detailed explanation of how the classification was obtained using Vinberg's second method.

1.3 Geometric technique

The work of Kraśkiewicz and Weyman relies on the use of the geometric technique of calculating syzygies. We provide here a brief overview of the technique applied to the context of our work. For more details, the reader can consult [20].

Let us fix a representation $G_0 \circlearrowright \mathfrak{g}_1$, arising from a pair (X_n, α_k); also, let $\mathcal{O}$ be an orbit of G_0 in $\mathfrak{g}_1$. Consider $\mathfrak{g}_1$ as an affine space $\mathbb{A}^N_{\mathbb{C}}$; every orbit closure $\overline{\mathcal{O}}$ in $\mathfrak{g}_1$ is an affine variety in $\mathbb{A}^N_{\mathbb{C}}$. Moreover, every orbit closure $\overline{\mathcal{O}}$ admits a desingularization Z which is the total space of a homogeneous vector bundle $\mathcal{S}$ on some homogeneous space G_0/P, for some parabolic subgroup P of G_0. The space $\mathbb{A}^N_{\mathbb{C}} \times G_0/P$ can be viewed as the total space of a trivial vector bundle $\mathcal{E}$ of rank N over G_0/P and $\mathcal{S}$ is a subbundle of $\mathcal{E}$. Altogether we have the following picture:

$$\begin{array}{ccc} Z & \hookrightarrow & \mathbb{A}^N_{\mathbb{C}} \times G_0/P \\ \downarrow{\scriptstyle q'} & & \downarrow{\scriptstyle q} \\ \overline{\mathcal{O}} & \hookrightarrow & \mathbb{A}^N_{\mathbb{C}} \end{array}$$

where q is the projection on $\mathbb{A}^N_{\mathbb{C}}$ and q' is its restriction to Z.

Let $A = \mathbb{C}[\mathbb{A}^N_{\mathbb{C}}] = \mathrm{Sym}(\mathfrak{g}_1^*)$; this is the polynomial ring over which we will carry out all computations. Also, introduce the vector bundle $\xi = (\mathcal{E}/\mathcal{S})^*$ on G_0/P. We can now state the basic theorem [20, Thm. 5.1.2].

Theorem 1.3.1. Define the graded free A-modules:

$$F_i := \bigoplus_{j \geqslant 0} H^j \left(G_0/P, \bigwedge^{i+j} \xi \right) \otimes_{\mathbb{C}} A(-i-j).$$

There exist minimal G_0-equivariant differentials

$$d_i \colon F_i \longrightarrow F_{i-1}$$

of degree 0 such that $F_\bullet$ is a complex of graded free A-modules with

$$H_{-i}(F_\bullet) = \mathcal{R}^i q_* \mathcal{O}_Z.$$

In particular, $F_\bullet$ is exact in positive degrees.

Recall that Z is a desingularization of $\overline{\mathcal{O}}$; in particular, the map $q' \colon Z \to \overline{\mathcal{O}}$ is a birational isomorphism. The next theorem [20, Thm. 5.1.3] gives a criterion for the complex $F_\bullet$ to be a free resolution of the coordinate ring of $\overline{\mathcal{O}}$.

Theorem 1.3.2. The following properties hold.

1. The module $q'_* \mathcal{O}_Z$ is the normalization of $\mathbb{C}[\overline{\mathcal{O}}]$.
2. If $\mathcal{R}^i q_* \mathcal{O}_Z = 0$ for $i > 0$, then $F_\bullet$ is a finite free resolution of the normalization of $\mathbb{C}[\overline{\mathcal{O}}]$ as an A-module.
3. If $\mathcal{R}^i q_* \mathcal{O}_Z = 0$ for $i > 0$ and $F_0 = A$, then $\overline{\mathcal{O}}$ is normal and has rational singularities.

The modules F_i are defined in terms of cohomology groups of the bundles $\bigwedge^{i+j} \xi$ on the homogeneous space G_0/P. These cohomology groups can be computed directly using Bott's algorithm when ξ is semisimple [20, Thm. 4.1.4]. When ξ is not semisimple, its structure becomes complicated so working with it directly is more difficult. However it is still possible to calculate the equivariant Euler characteristic of the bundles $\bigwedge^{i+j} \xi$. This was done by Kraśkiewicz and Weyman in [11]; their work provides the Hilbert functions of the normalization of the orbit closures together with an estimate for the shape of the minimal free resolution of their coordinate rings in terms of G_0-modules and equivariant maps.

Chapter 2

Computations

The expected resolution $F_\bullet$ of the coordinate ring of $\overline{\mathcal{O}}$ is constructed using the information provided by the equivariant Euler characteristic. As such it may be missing those syzygies the Euler characteristic was unable to detect due to cancellation; namely all those corresponding to an irreducible representation occurring in neighboring homological degrees and in the same homogeneous degree of the resolution. By constructing $F_\bullet$ explicitly in Macaulay2 (M2) [5], we can ensure it is the actual resolution for $\overline{\mathcal{O}}$.

In this chapter, we describe the type of computations that were carried out. All computations were run in M2. Although M2 can compute minimal free resolutions directly, this is generally impractical for many of the examples we consider, given the amount of computational resources and time needed for the algorithms to produce any result. Still, for some of the smaller examples, we were able to obtain resolutions directly or using the options `DegreeLimit` and `LengthLimit` to aid the computation.

2.1 The interactive method for syzygies

Let $F_\bullet$ be the expected resolution for $\overline{\mathcal{O}}$ and suppose that some differential $d_i\colon F_i \to F_{i-1}$ can be written explicitly as a matrix with entries in the polynomial ring A. Using M2 and the command `syz`, one can find the first syzygies of d_i. Since we expect these to coincide with the differential d_{i+1}, we use the expected degree of d_{i+1} as a degree bound with the option `DegreeLimit` to speed up the computation. This procedure can be iterated to recover the tail of the expected resolution:

$$\operatorname{coker} d_i \longleftarrow F_{i-1} \xleftarrow{d_i} F_i \longleftarrow T_\bullet$$

Similarly, transposing d_i and calculating syzygies gives the complex

$$H_\bullet \longrightarrow F_{i-1}^* \xrightarrow{d_i^*} F_i^* \longrightarrow \operatorname{coker} d_i^*$$

which can be dualized to obtain the head of the expected resolution. Splicing these two complexes together, we obtain the expected resolution in the form:

$$H_\bullet^* \longleftarrow F_{i-1} \xleftarrow{d_i} F_i \longleftarrow T_\bullet$$

Clearly, taking syzygies with degree bounds and dualizing may cause the resulting complex not to be exact; we prove exactness with the methods described in chapter 3. We refer to this method of constructing the expected resolution as the *interactive method* for calculating syzygies.

Remark 2.1.1. In most cases, given the decomposition into irreducible representations of the modules F_i, the differentials d_i are uniquely determined by Schur's lemma, up to a choice of scalars. However writing them explicitly is in general quite complicated. When the defining equations, i.e. the differential d_1, are not known, our choice of differential d_i to write explicitly falls on those matrices that are easier to describe (e.g. matrices with linear entries).

2.2 The cone procedure

When the orbit closure $\overline{\mathcal{O}}$ is not normal, the geometric technique returns the expected resolution of the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}})$, as an A-module. We have an exact sequence of A-modules:

$$0 \longrightarrow \mathbb{C}[\overline{\mathcal{O}}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})] \longrightarrow C \longrightarrow 0$$

Using the interactive method, we recover the expected resolution $F_\bullet$ for $\mathcal{N}(\overline{\mathcal{O}})$; in particular, $d_1\colon F_1 \to F_0$ is a minimal presentation of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})]$. As a G_0-representation, the module C is always obtained from $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})]$ by removing some irreducible representations. This implies that, in the appropriate bases for F_0 and F_1, a presentation of C is given by a map of free modules $d_1' : F_1' \to F_0'$ whose matrix is obtained from that of d_1 by dropping some rows and columns. This presentation can be used in M2 to construct a resolution $F_\bullet'$ for C (using the interactive method, if necessary). Moreover, the projection $\pi_0\colon F_0 \to F_0'$ lifts to a map of complexes $\tilde{\pi}\colon F_\bullet \to F_\bullet'$; this can be achieved explicitly in M2 with the command `extend` or via a step by step factorization with the command `//`. It is well known that the cone of $\tilde{\pi}$ is a (non necessarily minimal) free resolution of $\mathbb{C}[\overline{\mathcal{O}}]$; in M2, this is recovered with `cone`. Given the shift in homological degree introduced by the cone, $\mathbb{C}[\overline{\mathcal{O}}]$ is the degree one homology of the cone of $\tilde{\pi}$. Now $\mathbb{C}[\overline{\mathcal{O}}]$ can be minimized and resolved directly or using the interactive method. We call this the *cone procedure*.

Remark 2.2.1. A more efficient version of this technique comes from the realization that, because the homology of the cone of $\tilde{\pi}$ is concentrated in degree one, it is not necessary to construct the whole cone in M2. In fact it is enough to know the resolutions of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})]$ and C up to homological degree two and use them to construct the part of the cone needed to recover the homology in degree one. We will refer to this as the *truncated cone procedure*.

Remark 2.2.2. After the resolution of $\mathbb{C}[\overline{\mathcal{O}}]$ has been calculated in M2 using the cone procedure, its equivariant form can be recovered using the method described in appendix B. In particular, the term of the resolution in homological dimension zero is always a free module of rank one generated in homogeneous degree zero. This corresponds to a trivial representation for the group acting.

Chapter 3

Exactness

Once the expected resolution for the coordinate ring of (the normalization of) an orbit closure has been constructed in M2, we turn to the issue of proving it is exact.

3.1 The equivariant exactness criterion

Recall the exactness criterion of Buchsbaum and Eisenbud [2, Thm. 20.9]:

Theorem 3.1.1. Let A be a ring and

$$F_\bullet : \qquad F_0 \xleftarrow{d_1} F_1 \longleftarrow \dots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n$$

a complex of free A-modules. $F_\bullet$ is exact if and only if $\forall k = 1, \dots, n$:

1. $\operatorname{rank}(F_k) = \operatorname{rank}(d_k) + \operatorname{rank}(d_{k+1})$;
2. $\operatorname{depth}(I(d_k)) \geqslant k$ where $I(d_k)$ is the ideal of A generated by maximal non vanishing minors of d_k.

The map d_{n+1} is understood to be the zero map.

When A is our polynomial ring and $F_\bullet$ is the expected resolution of an orbit closure $\overline{\mathcal{O}}$, we have a more efficient way of proving exactness relying on the fact that $F_\bullet$ is G_0-equivariant.

Recall that G_0 acts on $\mathbb{A}^N_{\mathbb{C}}$, identified with $\mathfrak{g}_1$, with finitely many orbits $\mathcal{O}_0, \ldots, \mathcal{O}_t$; it follows immediately that the action has a dense orbit, also referred to as the generic orbit. Let p_j be a representative of the orbit $\mathcal{O}_j$ and let p be a representative of the generic orbit.

Lemma 3.1.2. Let $d : F \to F'$ be a non zero, G_0-equivariant, minimal map of graded free A-modules of finite rank.

$$\operatorname{depth}(I(d)) = \min\{\operatorname{codim}(\overline{\mathcal{O}}_j) \mid \operatorname{rank}(d|_{p_j}) < \operatorname{rank}(d|_p)\}.$$

Notice that the minimum on the right hand side is taken over a non empty set, since the rank of d at the origin is zero.

Proof. Denote $\mathcal{V}(I(d))$ the zero set in $\mathbb{A}^N_{\mathbb{C}}$ of the ideal $I(d)$ of A. We have

$$\operatorname{depth}(I(d)) = \operatorname{depth}(\sqrt{I(d)}) = \operatorname{codim}(\mathcal{V}(I(d)))$$

where the first equality holds because radicals preserve depth and the second equality follows from the fact the that the polynomial ring A is Cohen-Macaulay. By assumption, d is G_0-equivariant, hence $I(d)$ is G_0-equivariant and so is its zero set. Therefore we can write

$$\mathcal{V}(I(d)) = \bigcup_{\operatorname{rank}(d|_{p_j}) < \operatorname{rank}(d|_p)} \mathcal{O}_j$$

which is a finite union of G_0-orbits. Finally

$$\operatorname{codim}(\mathcal{V}(I(d))) = \min\{\operatorname{codim}(\overline{\mathcal{O}}_j) \mid \operatorname{rank}(d|_{p_j}) < \operatorname{rank}(d|_p)\}.$$

□

Proposition 3.1.3 (Equivariant exactness criterion)**.** Let $F_\bullet$ be a G_0-equivariant minimal complex of graded free A-modules and assume the differentials $d_i : F_i \to F_{i-1}$ are non zero. Then $F_\bullet$ is exact if and only if $\forall k = 1, \ldots, n$:

1. $\mathrm{rank}(F_k) = \mathrm{rank}(d_k|_p) + \mathrm{rank}(d_{k+1}|_p)$;

2. $\min\{\mathrm{codim}(\overline{\mathcal{O}}_j) \mid \mathrm{rank}(d_k|_{p_j}) < \mathrm{rank}(d_k|_p)\} \geqslant k$.

Proof. Condition 1 is equivalent to the first condition in the criterion of Buchsbaum and Eisenbud. This is because p is a representative of the generic orbit and therefore $\mathrm{rank}(d_k|_p) = \mathrm{rank}(d_k)$.

As for condition 2, this is equivalent to the second condition in the criterion of Buchsbaum and Eisenbud, because, by lemma 3.1.2, we have

$$\mathrm{depth}(I(d_k)) = \min\{\mathrm{codim}(\overline{\mathcal{O}}_j) \mid \mathrm{rank}(d_k|_{p_j}) < \mathrm{rank}(d_k|_p)\} \geqslant k.$$

□

3.2 Dual complexes and Cohen-Macaulay orbit closures

A useful feature of the equivariant exactness criterion is that it easily allows to check if the dual complex $F_\bullet^*$ is exact, providing us with the following.

Corollary 3.2.1. Under the hypotheses of proposition 3.1.3, suppose $\exists j_0 \in \{0, \ldots, t\}$ such that whenever $\mathrm{rank}(d_k|_{p_j}) < \mathrm{rank}(d_k|_p)$, for some k and j, we have:

- $\mathrm{rank}(d_k|_{p_{j_0}}) < \mathrm{rank}(d_k|_p)$;
- $\mathrm{codim}(\overline{\mathcal{O}}_{j_0}) \leqslant \mathrm{codim}(\overline{\mathcal{O}}_j)$.

Then $F_\bullet$ is exact if and only if $F_\bullet^*$ is exact.

Remark 3.2.2. The conditions in the corollary can be simply restated by saying that the ranks of the differentials d_k drop simultaneously at the same orbit $\overline{\mathcal{O}}_{j_0}$.

Proof. First observe that $\mathrm{rank}(d_k^*|_{p_j}) = \mathrm{rank}(d_k|_{p_j})$. Now apply the equivariant exactness criterion. Condition 1 is trivially satisfied. Condition 2 together with the hypothesis implies

$$\mathrm{codim}(\overline{\mathcal{O}}_{j_0}) = \min\{\mathrm{codim}(\overline{\mathcal{O}}_j) \mid \mathrm{rank}(d_k|_{p_j}) < \mathrm{rank}(d_k|_p)\} \geqslant k$$

for all k. Since the left hand side is independent of k, condition 2 must be satisfied for $F_\bullet$ and $F_\bullet^*$ at the same time. □

The above corollary implies that when $F_\bullet$ is exact and the ranks of the differentials d_k drop simultaneously at the same orbit, $F_\bullet$ resolves a perfect module. In particular, if $F_\bullet$ resolves the coordinate ring of some orbit closure $\overline{\mathcal{O}}$, then $\overline{\mathcal{O}}$ is Cohen-Macaulay.

Remark 3.2.3. As detailed in the discussion of the interactive method 2.1, the head $H_\bullet$ of the expected resolution is obtained by transposing the matrix of a differential d_i, resolving its cokernel and then dualizing back. It was noted then that taking duals could affect exactness. However, it follows immediately from our previous corollary, that if $\overline{\mathcal{O}}$ is Cohen-Macaulay both $H_\bullet$ and $H_\bullet^*$ are exact.

Chapter 4

Equations of orbit closures

Once we establish that the expected resolution $F_\bullet$ for an orbit closure $\overline{\mathcal{O}}$ is exact, the entries of the first differential $d_1\colon F_1 \to F_0$ generate an ideal I of A whose zero set $\mathcal{V}(I)$ is precisely $\overline{\mathcal{O}}$. In this chapter, we address the issue of determining if I is a radical ideal. We also describe a simple criterion to establish inclusion and singularity of the orbit closures.

4.1 The coordinate ring of an orbit closure

The generators of the ideal I provide equations for the variety $\overline{\mathcal{O}}$. There is no guarantee however that I is radical; equivalently, A/I need not be reduced, hence it may not be the coordinate ring of $\overline{\mathcal{O}}$.

We outline here the method used to show that the ring $R = A/I$ is indeed reduced. The proof relies on the following characterization of reduced rings [2, Ex. 11.10] which is an analogue of Serre's criterion for normality.

Proposition 4.1.1. A Noetherian ring R is reduced if and only if it satisfies:

(R_0): the localization of R at each prime of height 0 is regular;

(S_1): all primes associated to zero have height 0.

We address the condition (S_1) first. In our case, $\overline{\mathcal{O}}$ is always irreducible since it is an orbit closure for an irreducible group G_0. This implies that the ideal I has a unique minimal prime, namely $\sqrt{I}$. Therefore the condition (S_1) is equivalent to $\sqrt{I}$ being the only associated prime of I. In particular, I will have no embedded primes.

To prove that $\sqrt{I}$ is the only associated prime of I, we adapt a result from [2, Cor. 20.14].

Lemma 4.1.2. Let

$$F_\bullet : \qquad F_0 \xleftarrow{d_1} F_1 \longleftarrow \dots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n \longleftarrow 0$$

be a free resolution of $R = A/I$ as an A-module and suppose $\operatorname{depth}_A(I) = c$. We have $\operatorname{depth}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}) = c$ for all associated primes $\mathfrak{p}$ of I if and only if $\operatorname{depth}(I(d_k)) > k$ for all $k > c$, where $I(d_k)$ is the ideal of A generated by the maximal non vanishing minors of d_k.

The following result provides a simple test for the condition (S_1) in all cases we consider.

Proposition 4.1.3. Let I be an ideal in the polynomial ring A such that the zero locus $\mathcal{V}(I)$ of I is irreducible of codimension c. Let

$$F_\bullet : \qquad F_0 \xleftarrow{d_1} F_1 \longleftarrow \dots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n \longleftarrow 0$$

be a free resolution of $R = A/I$ as an A-module. If $\operatorname{depth}(I(d_k)) > k$ for all $k > c$, then $\sqrt{I}$ is the only associated prime of I and R satisfies (S_1).

Proof. First observe that

$$c = \operatorname{codim}(\mathcal{V}(I)) = \operatorname{depth}_A(\sqrt{I}) = \operatorname{depth}_A(I),$$

because the ring A is Cohen-Macaulay and taking radicals preserve depth.

Now let $\mathfrak{p}$ be an associated prime of I. By lemma 4.1.2, we have $\operatorname{depth}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}) = c$. Since $\sqrt{I}$ is the unique minimal prime of I, the inclusion $\sqrt{I} \subseteq \mathfrak{p}$ holds. We deduce that

$$c = \operatorname{depth}_A(\sqrt{I}) \leqslant \operatorname{depth}_A(\mathfrak{p}) \leqslant \operatorname{depth}_{A_\mathfrak{p}}(\mathfrak{p}A_\mathfrak{p}) = c,$$

where the last inequality holds because passing to the localization preserves regular sequences. Since A is Cohen-Macaulay, it follows that

$$\text{height}(\mathfrak{p}) = \text{depth}_A(\mathfrak{p}) = c.$$

But $\sqrt{I} \subseteq \mathfrak{p}$ and both ideals are primes of height c. Therefore $\mathfrak{p} = \sqrt{I}$. □

Remark 4.1.4. The test given in 4.1.3 is quite easy to apply in practice. The (co)dimension of each orbit closure is known a priori. Moreover, as observed in the proof of proposition 3.1.3,

$$\text{depth}(I(d_k)) = \min\{\text{codim}(\overline{\mathcal{O}}_j) \mid \text{rank}(d_k|_{p_j}) < \text{rank}(d_k|_p)\},$$

where p_j is a representative of the orbit $\mathcal{O}_j$ and p is a representative of the dense orbit.

We now turn to the condition (R_0) in 4.1.1.

Proposition 4.1.5. Let I be an ideal in the polynomial ring A with zero locus $\mathcal{V}(I)$ of codimension c. Assume also that $\mathcal{V}(I)$ is irreducible and that $\sqrt{I}$ is the only associated prime of I. Denote J the Jacobian matrix for a set of generators of I. If there exists a point $x \in \mathcal{V}(I)$ such that $\text{rank}(J|_x) = c$, then the ring $R = A/I$ satisfies the condition (R_0) and is therefore reduced.

Proof. Let $\mathfrak{m}$ be the maximal ideal of A corresponding to the point $x \in \mathcal{V}(I)$ and let $\mathfrak{p} = \sqrt{I}$. We denote $\overline{\mathfrak{m}}$ and $\overline{\mathfrak{p}}$ respectively, the images of $\mathfrak{m}$ and $\mathfrak{p}$ under the canonical projection $A \to A/I = R$. Clearly R satisfies (S_1), given the hypothesis on the associated primes of I. To prove R satisfies (R_0) we must show that the local ring $R_{\overline{\mathfrak{p}}}$ is regular, since $\overline{\mathfrak{p}}$ is the only height 0 prime in R.

First we observe that the ring $R_{\overline{\mathfrak{m}}}$ is regular by the Jacobian criterion [2, Th. 16.19]. By the transitivity of localization

$$R_{\overline{\mathfrak{p}}} \cong (R_{\overline{\mathfrak{m}}})_{\overline{\mathfrak{p}}R_{\overline{\mathfrak{m}}}}.$$

The localization of a regular local ring at a prime ideal is regular. Hence $R_{\overline{\mathfrak{p}}}$ is regular. □

Remark 4.1.6. In practice, for an orbit closure $\overline{\mathcal{O}} = \mathcal{V}(I)$, we will check that the rank of the Jacobian matrix of I is equal to $\text{codim}(\overline{\mathcal{O}})$ at a representative x of $\mathcal{O}$. Indeed the point x is smooth in $\overline{\mathcal{O}}$. This is because the singular locus of $\overline{\mathcal{O}}$ is equivariant; therefore it must be an orbit closure of codimension at least one in $\overline{\mathcal{O}}$ and cannot contain x.

Remark 4.1.7. A Cohen-Macaulay ring R is generically reduced (i.e. its localization at each minimal prime is reduced) if and only if it is reduced [2, Ex. 18.9]. Notice how generically reduced is precisely the condition (R_0). In fact, if our ring $R = A/I$ is Cohen-Macaulay, then it automatically satisfies (S_1). Therefore it is enough to check (R_0) as outlined above.

4.2 Inclusions and singular loci of orbit closures

One problem that can be answered easily once we have equations for the orbit closures is to determine how they sit one inside the other. To be more precise, we introduce the degeneration partial order by setting $\mathcal{O}_i \leqslant \mathcal{O}_j$ if and only if $\mathcal{O}_i \subseteq \overline{\mathcal{O}}_j$. Because we have finitely many orbits, the entire picture can be described by checking if the equations of $\overline{\mathcal{O}}_j$ vanish at the representative p_i of $\mathcal{O}_i$. This can be achieved conveniently in M2.

Remark 4.2.1. In a few cases of non normal orbit closures, the (truncated) cone procedure 2.2 did not return a result in a reasonable amount of time. In these cases, we did not obtain the defining equations of the orbit closure. However we can still determine which orbits are contained in the orbit closure. In fact, the vanishing locus of the defining ideal of an orbit closure $\overline{\mathcal{O}}_j$ is the support of the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_j]$, which coincides with the support of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_j)]$. This follows from the fact that $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_j)]$ is the integral closure of $\mathbb{C}[\overline{\mathcal{O}}_j]$ and therefore it is a finitely generated $\mathbb{C}[\overline{\mathcal{O}}_j]$-module. Hence, instead of checking the vanishing of the defining equations, we check the rank of the presentation of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_j)]$ at the representative of the orbit $\mathcal{O}_i$. If this rank is less than the rank on the dense orbit, then $\mathcal{O}_i \subseteq \overline{\mathcal{O}}_j$.

Next we can use the equations of an orbit closure to construct a Jacobian matrix and apply the Jacobian criterion to determine the singular locus of the orbit closure [6, p. 31]. Once again, it is enough to evaluate the rank of the Jacobian matrix at finitely many points, the points being representatives of the orbits.

Remark 4.2.2. For a non normal orbit closure we have:

$$0 \longrightarrow \mathbb{C}[\overline{\mathcal{O}}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})] \longrightarrow C \longrightarrow 0$$

Since $\overline{\mathcal{O}}$ is an affine variety in $\mathbb{A}^N_{\mathbb{C}}$, the middle term in the sequence is simply the integral closure of $\mathbb{C}[\overline{\mathcal{O}}]$. If $\mathfrak{p}$ is a prime ideal in $A = \mathbb{C}[\mathbb{A}^N_{\mathbb{C}}]$, then localizing the sequence above at $\mathfrak{p}$, we obtain a new short exact sequence:

$$0 \longrightarrow \mathbb{C}[\overline{\mathcal{O}}]_{\mathfrak{p}} \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})]_{\mathfrak{p}} \longrightarrow C_{\mathfrak{p}} \longrightarrow 0$$

where, by the properties of localization, the middle term is still the integral closure of the term on the left. In particular, we deduce that $\mathbb{C}[\overline{\mathcal{O}}]_{\mathfrak{p}}$ is integrally closed if and only if $C_{\mathfrak{p}} = 0$, i.e. if and only if $\mathfrak{p}$ is not in the support of C. We can express this by saying that the support of C is the non normal locus of $\overline{\mathcal{O}}$.

In some cases of non normal orbit closures, when the defining equations of the orbit closure could not be recovered explicitly in M2, the observation above can be used to determine the singular locus of an orbit closure. In fact, the non normal locus of a variety V is contained inside the singular locus of V (since every regular local ring is integrally closed [2]).

The information on containment and singularity of the orbit closures is presented in a table with rows labeled by the orbits and columns labeled by their closures. The cell corresponding to a row $\mathcal{O}_i$ and a column $\overline{\mathcal{O}}_j$ can be empty, meaning the points of the orbit $\mathcal{O}_i$ are not contained in the orbit closure $\overline{\mathcal{O}}_j$, or it can contain the letters 'ns' (respectively 's') to indicate that the points of $\mathcal{O}_i$ are non singular (respectively singular) in the orbit closure $\overline{\mathcal{O}}_j$.

4.3 The degenerate orbits

Let (X_n, x) be a Dynkin diagram with a distinguished node. The Lie algebra $\mathfrak{g}$ of type X_n has a grading

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

induced by the choice of the distinguished node x, as described in chapter 1. Let $\mathcal{O}$ be an orbit for the action $G_0 \circlearrowright \mathfrak{g}_1$. Suppose there is a node $y \neq x$ in X_n such that $\mathcal{O} \cap \mathfrak{g}'_1 \neq 0$, where $\mathfrak{g}'$ is the graded subalgebra of $\mathfrak{g}$ corresponding to the subdiagram $(X_n \backslash \{x\}, y)$. Then we say the orbit $\mathcal{O}$ is degenerate. This means that $\mathcal{O}$ comes from an orbit $\mathcal{O}'$ that occurs in the case of a smaller Dynkin diagram. The paper of Kraśkiewicz and Weyman [11] describes a method to obtain the free resolution for the coordinate ring of $\overline{\mathcal{O}}$ by reducing to the case of $\mathcal{O}'$.

In practice, many of the orbit closures of degenerate orbits that we encountered are well known varieties. When possible, we describe the defining equations directly. For some cases, we can obtain the defining ideal by polarizing the equations for the closure of the smaller orbit with respect to the inclusion $\mathfrak{g}'_1 \hookrightarrow \mathfrak{g}_1$, and adding (if they are not already contained in the ideal) the equations of the so-called generic degenerate orbit. The latter is an orbit closure in $\mathfrak{g}_1$ arising from an orbit closure in $\mathfrak{g}'_1$ whose closure is all of $\mathfrak{g}'_1$; it will be indicated on a case by case basis, when needed.

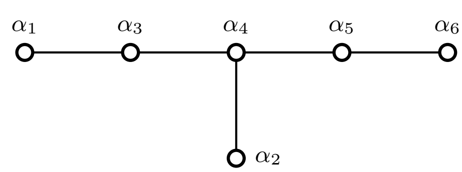
Chapter 5

An example

We provide here an example to better illustrate the techniques described in the previous chapters. This is an expanded version of the case discussed in 6.3.1. We chose this particular example because it is small but already quite interesting (the orbit closure is non normal and non Cohen-Macaulay).

5.1 The representation

Consider the Dynkin diagram of type E_6. Using the conventions adopted in Bourbaki [1], we label the nodes of the diagram $\alpha_1, \ldots, \alpha_6$. We select α_3 to be the distinguished node, so we will be working in the case (E_6, α_3).



The simple Lie algebra $\mathfrak{g}$ of type E_6 has the graded decomposition:

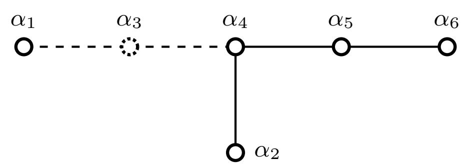
$$\mathfrak{g} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$$

where:

- $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_5(\mathbb{C})$,
- $\mathfrak{g}_1 = \mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^5$,
- $\mathfrak{g}_2 = \bigwedge^2 \mathbb{C}^2 \otimes \bigwedge^4 \mathbb{C}^5$,

and the components $\mathfrak{g}_{-i}$ are dual to the $\mathfrak{g}_i$ with respect to the Killing form of $\mathfrak{g}$ (see [11] for more details). Our representation will be the vector space $\mathfrak{g}_1$, with the action of the adjoint group of the Lie subalgebra $\mathfrak{g}_0$, i.e. the group $\mathbb{C}^\times \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_5(\mathbb{C})$.

There is also a nice rule of thumb to recover the representation directly from the diagram. Start by removing the distinguished node.



The group is the product of the complex simple Lie groups of the connected components of the new diagram, with an additional $\mathbb{C}^\times$ factor. In our case, we see a component of type A_1 and one of type A_4; hence the group is $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_5(\mathbb{C}) \times \mathbb{C}^\times$. In addition, each factor corresponding to a connected component has a fundamental representation corresponding to the node of the component which was connected to the distinguished node in the original diagram. The representation we are interested in is the tensor product of those fundamental representations. In our case, the distinguished node α_3 touched the component of type A_1 at its only node and the corresponding fundamental representation is $\bigwedge^1 \mathbb{C}^2 \cong \mathbb{C}^2$; similarly, the distinguished node touched the component of type A_4 at its second node and the corresponding fundamental representation is $\bigwedge^2 \mathbb{C}^5$. Thus the representation is $\mathbb{C}^2 \otimes \bigwedge^2 \mathbb{C}^5$.

5.2 The orbits

For the rest of this example, we set $E = \mathbb{C}^2$, with basis $\{e_1, e_2\}$, and $F = \mathbb{C}^5$, with basis $\{f_1, \dots, f_5\}$ (dual basis elements will be denoted with a star). Our representation is $E \otimes \bigwedge^2 F$

and the group acting is $G = \mathrm{SL}(E) \times \mathrm{SL}(F) \times \mathbb{C}^\times$.

With respect to the maximal tori of diagonal matrices in the factors of G, the tensors $e_a \otimes f_i \wedge f_j$ form a basis of weight vectors for the representation. The polynomial functions on $E \otimes \bigwedge^2 F$, regarded as an affine space, belong inside the ring

$$A = \mathrm{Sym}\left(E^* \otimes \bigwedge^2 F^*\right) = \mathbb{C}[x_{a;ij} \mid a = 1, 2; 1 \leqslant i < j \leqslant 5].$$

Here the variable $x_{a;ij}$ is the dual of the tensor $e_a \otimes f_i \wedge f_j$ inside $E^* \otimes \bigwedge^2 F^* = A_1$. The ring A is the polynomial ring we will use in M2 to carry out our computations.

Thanks to the work of Vinberg, we know the representation has eight orbits, $\mathcal{O}_0, \ldots, \mathcal{O}_7$. In the following table, we give the dimension of the Zariski closure of each orbit followed by a representative of the orbit. This table and the other similar ones appearing in this work are from [11].

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	8	$x_{1;12} = 1$
$\mathcal{O}_2$	11	$x_{1;12} = x_{1;34} = 1$
$\mathcal{O}_3$	12	$x_{1;12} = x_{2;13} = 1$
$\mathcal{O}_4$	15	$x_{1;12} = x_{1;34} = x_{2;13} = 1$
$\mathcal{O}_5$	16	$x_{1;12} = x_{2;34} = 1$
$\mathcal{O}_6$	18	$x_{1;12} = x_{2;34} = x_{1;35} = 1$
$\mathcal{O}_7$	20	$x_{1;12} = x_{2;34} = x_{1;35} = x_{2;15} = 1$

To obtain a representative for a given orbit, set all the listed variables to the corresponding value and all the other ones to 0. For example, a representative of the orbit $\mathcal{O}_1$ is the point with $x_{1;12} = 1$ and all other coordinates equal to 0; similarly, a representative of the orbit $\mathcal{O}_6$ has $x_{1;12} = x_{2;34} = x_{1;35} = 1$ and all other coordinates equal to 0. The general convention adopted here is to denote with $\mathcal{O}_0$ the origin and with the largest index the dense orbit, i.e. the orbit whose Zariski closure is the entire representation.

5.3 An orbit closure

We consider $\overline{\mathcal{O}}_6$, the Zariski closure of the orbit $\mathcal{O}_6$ from the previous section. This orbit closure is non normal and the expected resolution of Kraśkiewicz and Weyman for the coordinate ring of its normalization $\mathcal{N}(\overline{\mathcal{O}}_6)$ is

$$A \oplus \left(\bigwedge^2 E^* \otimes \bigwedge^4 F^* \otimes A(-2) \right) \leftarrow \mathbb{S}_{(2,1)} E^* \otimes \mathbb{S}_{(2,1,1,1,1)} F^* \otimes A(-3) \leftarrow$$
$$\leftarrow \mathbb{S}_{(4,1)} E^* \otimes \mathbb{S}_{(2,2,2,2,2)} F^* \otimes A(-5) \leftarrow 0$$

The free A-modules in the complex above are written as direct sums of representations of G tensored (over $\mathbb{C}$) with a copy of A generated in the appropriate degree. Hence, calculating the dimensions over $\mathbb{C}$ of the representations in the complex, we can write the Betti table for the expected resolution.

	0	1	2
total:	6	10	4
0:	1	.	.
1:	.	.	.
2:	5	10	.
3:	.	.	4

Eventually, we would like to recover the minimal free resolution of the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_6]$. Since $\overline{\mathcal{O}}_6$ is non normal, we observe that we have the following diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & A \oplus A(-2)^5 & \longrightarrow & A(-2)^5 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{C}[\overline{\mathcal{O}}_6] & \hookrightarrow & \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)] & \longrightarrow & C(6) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Because $\mathbb{C}[\overline{\mathcal{O}}_6]$ is a quotient of A and $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$ is a quotient of $A \oplus A(-2)^5$, the A summand in degree zero must split off from the generators of the cokernel $C(6)$ for the inclusion $\mathbb{C}[\overline{\mathcal{O}}_6] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$. Moreover $C(6)$ is a G-module so, from the expected resolution of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$, we get a presentation for $C(6)$ in the form

$$\bigwedge^2 E^* \otimes \bigwedge^4 F^* \otimes A(-2) \leftarrow \mathbb{S}_{(2,1)} E^* \otimes \mathbb{S}_{(2,1,1,1,1)} F^* \otimes A(-3)$$

This in turn can be used with the cone procedure described in 2.2 to recover the minimal free resolution of $\mathbb{C}[\overline{\mathcal{O}}_6]$.

5.4 An explicit differential

We begin from d_2, the last differential in the expected resolution of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$, and give an explicit realization. First notice that, up to powers of the determinant representations in the domain and codomain, our map can be thought of as

$$d_2 : \mathbb{S}_3 E^* \otimes A(-5) \longrightarrow E^* \otimes \bigwedge^4 F \otimes A(-3).$$

Moreover, the domain is generated in degree 5, hence we can restrict to that graded component and define a map

$$\mathbb{S}_3 E^* \longrightarrow E^* \otimes \bigwedge^4 F \otimes A_2.$$

One can observe (with the use of specialized software like LiE [17], or sometimes by hand) that the irreducible representation $\mathbb{S}_3 E^*$ appears with multiplicity one in the decomposition of the tensor product in the codomain. It follows from Schur's lemma [3, Lemma 1.7] that, up to scalars, there is only one non zero map between the given representations.

To construct the map, we use the equivariant maps described in A. We operate in two steps. For the first one, we take the map

$$\mathbb{S}_3 E^* \xrightarrow{\ \Delta\ } \mathbb{S}_2 E^* \otimes E^* \xrightarrow{tr^{(4)}} \mathbb{S}_2 E^* \otimes E^* \otimes \bigwedge^4 F \otimes \bigwedge^4 F^*$$

which sends the basis element $e_i^* e_j^* e_k^*$ of $\mathbb{S}_3 E^*$ to

$$(e_i^* \otimes e_j^* e_k^* + e_j^* \otimes e_i^* e_k^* + e_k^* \otimes e_i^* e_j^*)\otimes$$
$$\otimes \left(\sum_{1 \leqslant u_1 < u_2 < u_3 < u_4 \leqslant 5} f_{u_1} \wedge f_{u_2} \wedge f_{u_3} \wedge f_{u_4} \otimes f_{u_1}^* \wedge f_{u_2}^* \wedge f_{u_3}^* \wedge f_{u_4}^* \right).$$

The second step operates on $\mathbb{S}_2 E^* \otimes E^* \otimes \bigwedge^4 F \otimes \bigwedge^4 F^*$ via the identity map on the second and third factor, while on the first and fourth it operates via the map

$$\begin{array}{c} \mathbb{S}_2 E^* \otimes \bigwedge^4 F^* \\ \downarrow \Delta \\ \mathbb{S}_2 E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\ \downarrow m_{2,3} \\ \mathbb{S}_2 E^* \otimes \mathbb{S}_2(\bigwedge^2 F^*) \\ \big\downarrow \\ A_2 \end{array}$$

The embedding into A_2 takes the basis element $e_i^* e_j^* \otimes (f_{u_1}^* \wedge f_{u_2}^*)(f_{u_3}^* \wedge f_{u_4}^*)$ to

$$\frac{1}{2}(x_{i;u_1u_2} x_{j;u_3u_4} + x_{j;u_1u_2} x_{i;u_3u_4}),$$

and the whole second step will send the basis element $e_i^* e_j^* \otimes f_{u_1}^* \wedge f_{u_2}^* \wedge f_{u_3}^* \wedge f_{u_4}^*$ to

$$x_{i;u_1u_2} x_{j;u_3u_4} + x_{j;u_1u_2} x_{i;u_3u_4} - x_{i;u_1u_3} x_{j;u_2u_4}$$
$$-x_{j;u_1u_3} x_{i;u_2u_4} + x_{i;u_1u_4} x_{j;u_2u_3} + x_{j;u_1u_4} x_{i;u_2u_3}.$$

Applying the two steps to a basis of $\mathbb{S}_3 E^*$, we can write the matrix for the differential d_2.

Remark 5.4.1. In the course of preparing this work, the equivariant maps constructed explicitly were often written using code for M2. However, the code was written ad hoc for each case and an attempt to code a more general algorithm that would work for all such maps was discarded early on.

5.5 The resolution

To recover the first differential we use the interactive method 2.1. Explicitly, we compute syzygies for the transpose of the matrix of d_2. Using the command `syz` in M2, we can provide a degree bound through the option `DegreeLimit`. This needs to be set to the appropriate value observed from the Betti table of the expected resolution (see 5.3); in our example, we expect d_1 to have entries of degree 1 and 3. The resulting matrix needs to be transposed to give a matrix for d_1.

Remark 5.5.1. For this particular example, it would be sufficient to resolve the cokernel of the transpose of d_2 using the command `res`, even without degree bounds. However, we proceeded as with the more computationally intensive cases, in order to fully illustrate the interactive method.

At this point, we have matrices for all the differentials in the expected resolution; we are guaranteed that they form a chain complex since they were obtained via iterated syzygies. To prove exactness we use the equivariant exactness criterion (proposition 3.1.3). The following table contains columns for each differential and rows for each orbit. In the cell at the intersection of a given row and column, we put the rank of the differential for the column evaluated at the representative for the orbit in the given row. The representatives for the orbits are listed in 5.2.

orbit	rank(d_1)	rank(d_2)
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	2	0
$\mathcal{O}_2$	4	2
$\mathcal{O}_3$	3	0
$\mathcal{O}_4$	4	2
$\mathcal{O}_5$	4	2
$\mathcal{O}_6$	5	3
$\mathcal{O}_7$	6	4

The maximal ranks are achieved on the dense orbit $\mathcal{O}_7$. We can see that, in the last row, the ranks of two consecutive differentials add up to the rank of the free module in the middle; this is condition 1 in the equivariant exactness criterion. The second condition is checked by observing that the ranks drop at the orbit $\mathcal{O}_6$ whose closure has codimension 2. Hence the complex is exact.

5.6 The coordinate ring

As outlined in 5.3, we can now take our first differential d_1 and delete the only row with entries of degree 3. The matrix thus obtained is a presentation for the cokernel $C(6)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_6] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$. We can easily find a minimal free resolution for $C(6)$, either directly with `res` or with the interactive method followed by an application of the equivariant exactness criterion. The projection from $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$ onto $C(6)$, lifts to a map of complexes between the resolutions of those two modules; in M2, this is obtained using `extend`. Finally, the `cone` command applied to this map returns a free resolution whose homology in homological degree 1 is A/I, where I is an ideal of A with zero locus $\overline{\mathcal{O}}_6$. We give here the Betti table for the minimal free resolution of A/I:

	0	1	2	3
total:	1	10	10	1
0:	1	.	.	.
1:	.	.	.	.
2:	.	.	.	.
3:	.	.	.	.
4:	.	.	.	.
5:	.	10	10	.
6:	.	.	.	.
7:	.	.	.	1

The following is the table with the ranks of the differentials in the resolution of A/I evaluated at the representatives of the orbits:

orbit	rank(d_1)	rank(d_2)	rank(d_3)
$\mathcal{O}_0$	0	0	0
$\mathcal{O}_1$	0	3	0
$\mathcal{O}_2$	0	5	0
$\mathcal{O}_3$	0	5	0
$\mathcal{O}_4$	0	6	0
$\mathcal{O}_5$	0	6	0
$\mathcal{O}_6$	0	8	1
$\mathcal{O}_7$	1	9	1

Once again this table can be used with the equivariant exactness criterion 3.1.3 to prove the complex is indeed exact. Next we use the table to test the ring A/I satisfies the condition (S_1) as outlined in proposition 4.1.3. The orbit closure $\overline{\mathcal{O}}_6$ has codimension 2. From the table, we see that $\operatorname{rank}(d_3)$ drops at the orbit closure $\overline{\mathcal{O}}_5$, which has codimension 4. This shows $\operatorname{depth}(I(d_3)) = 4 > 3$ and therefore A/I satisfies (S_1).

We can recover generators of I by taking the entries of the first differential of the resolution of A/I. These equations can be used to determine which orbit closures are contained in $\overline{\mathcal{O}}_6$. The table above shows that d_1 vanishes at all orbits except for $\mathcal{O}_7$, thus $\overline{\mathcal{O}}_6$ contains all orbit closures except $\overline{\mathcal{O}}_7$.

To ensure $A/I \cong \mathbb{C}[\overline{\mathcal{O}}_6]$, we still need to check that A/I satisfies (R_0). We proceed as in proposition 4.1.5. First we form the Jacobian matrix using the generators of I in the matrix of d_1. Then we evaluate the rank of the Jacobian matrix at the representatives of all orbits contained in $\overline{\mathcal{O}}_6$. We record the results in the following table:

orbit	$\operatorname{rank}(J)$
$\mathcal{O}_0$	0
$\mathcal{O}_1$	0
$\mathcal{O}_2$	0
$\mathcal{O}_3$	0
$\mathcal{O}_4$	0
$\mathcal{O}_5$	0
$\mathcal{O}_6$	2

Since $\overline{\mathcal{O}}_6$ has codimension 2, we deduce that it is singular at all its points except those belonging to the orbit $\mathcal{O}_6$. This finally shows that A/I is indeed the coordinate ring of $\overline{\mathcal{O}}_6$.

We conclude the discussion of this case by observing that $\overline{\mathcal{O}}_6$ is not Cohen-Macaulay. In fact it has codimension 2 but its coordinate ring has projective dimension 3.

Remark 5.6.1. The equivariant form for the resolution of $\mathbb{C}[\overline{\mathcal{O}}_6]$ can easily be recovered using the technique outlined in appendix B:

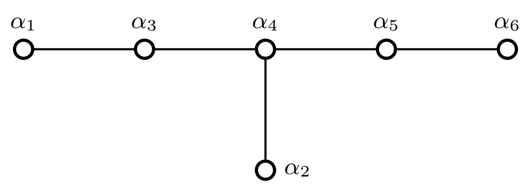
$$A \leftarrow \mathbb{S}_{(3,3)}\, E^* \otimes \mathbb{S}_{(3,3,2,2,2)}\, F^* \otimes A(-6) \leftarrow$$

$$\leftarrow \mathbb{S}_{(4,3)}\, E^* \otimes \mathbb{S}_{(3,3,3,3,2)}\, F^* \otimes A(-7) \leftarrow \mathbb{S}_{(5,5)}\, E^* \otimes \mathbb{S}_{(4,4,4,4,4)}\, F^* \otimes A(-10) \leftarrow 0$$

Chapter 6

Representations of type E_6

In this chapter, we will analyze the cases corresponding to gradings on the simple Lie algebra of type E_6. Each case corresponds to the choice of a distinguished node on the Dynkin diagram for E_6. The nodes are numbered according to the conventions in Bourbaki [1].



Given the symmetry in the diagram, it is enough to consider the cases with $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ as the distinguished nodes. The case (E_6, α_5) is equivalent to (E_6, α_3), while (E_6, α_6) is equivalent to (E_6, α_1).

6.1 The case (E_6, α_1)

The representation is $V(\omega_4, D_5)$, the half spinor representation for the group Spin(10). It has dimension 16 and weight vectors of the form $(\lambda_1, \ldots, \lambda_5)$, where $\lambda_i = \pm\frac{1}{2}$, with an even number of negative coordinates. Each weight is labeled by $[I]$, where

$$I = \left\{ i \in \{1,2,3,4,5\} \middle| \lambda_i = -\tfrac{1}{2} \right\}.$$

The corresponding polynomial ring is

$$A = \mathbb{C}[x_{\varnothing}, x_{ab}, x_{ijkl} | 1 \leqslant a < b \leqslant 5, 1 \leqslant i < j < k < l \leqslant 5].$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	11	$x_{\varnothing} = 1$
$\mathcal{O}_2$	16	$x_{\varnothing} = x_{1234} = 1$

All the orbit closures are normal, Cohen-Macaulay and have rational singularities. Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$
$\mathcal{O}_0$	ns	s	ns
$\mathcal{O}_1$		ns	ns
$\mathcal{O}_2$			ns

6.1.1 The orbit $\mathcal{O}_1$

The variety $\overline{\mathcal{O}}_1$ is the closure of the highest weight vector orbit and it is known as the variety of pure spinors. The defining equations were described by Manivel in [15] and can be resolved directly in M2. The Betti table for the resolution is

	0	1	2	3	4	5
total:	1	10	16	16	10	1
0:	1	.	.	.	.	.
1:	.	10	16	.	.	.
2:	.	.	.	16	10	.
3:	.	.	.	.	.	1

This orbit closure is Gorenstein.

6.2 The case (E_6, α_2)

The representation is $\bigwedge^3 F$, where $F = \mathbb{C}^6$; the group acting is $\mathrm{GL}(F)$. The weights of $\bigwedge^3 F$ are of the form $\epsilon_i + \epsilon_j + \epsilon_k$ for $1 \leqslant i < j < k \leqslant 6$. We label the corresponding weight vector by $[ijk]$ where $1 \leqslant i < j < k \leqslant 6$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{ijk} | 1 \leqslant i < j < k \leqslant 6] = \mathrm{Sym}\left(\bigwedge^3 F^*\right).$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	10	$x_{123} = 1$
$\mathcal{O}_2$	15	$x_{123} = x_{145} = 1$
$\mathcal{O}_3$	19	$x_{123} = x_{145} = x_{246} = 1$
$\mathcal{O}_4$	20	$x_{123} = x_{456} = 1$

All the orbit closures are normal, Cohen-Macaulay, Gorenstein and have rational singularities. Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$	$\overline{\mathcal{O}}_3$	$\overline{\mathcal{O}}_4$
$\mathcal{O}_0$	ns	s	s	s	ns
$\mathcal{O}_1$		ns	s	s	ns
$\mathcal{O}_2$			ns	s	ns
$\mathcal{O}_3$				ns	ns
$\mathcal{O}_4$					ns

When describing the resolutions over A we write simply (λ) for the Schur module $\mathbb{S}_\lambda F^* \otimes A(-|\lambda|/3)$.

6.2.1 The orbit $\mathcal{O}_3$

The orbit closure $\overline{\mathcal{O}}_3$ is a hypersurface defined by an invariant of degree 4 which can be obtained as follows:

$$\begin{array}{c}
\bigwedge^6 F^* \otimes \bigwedge^6 F^* \\
\downarrow \Delta\otimes\Delta \\
\bigwedge^3 F^* \otimes \bigwedge^2 F^* \otimes F^* \otimes F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^3 F^* \\
\downarrow m_{2,4}\otimes m_{3,5} \\
\bigwedge^3 F^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 F^* \\
\hookdownarrow \\
A_4
\end{array}$$

6.2.2 The orbit $\mathcal{O}_2$

The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_2]$ is

$$(0^6) \longleftarrow (2^3, 1^3) \longleftarrow (3, 2^4, 1) \longleftarrow (4, 3^4, 2) \longleftarrow (4^3, 3^3) \longleftarrow (5^6) \longleftarrow 0$$

The differential d_2 was written explicitly as the map

$$F \otimes F^* \xrightarrow{tr^{(2)}} F \otimes F^* \otimes \bigwedge^2 F \otimes \bigwedge^2 F^* \xrightarrow{m_{13}\otimes m_{24}} \bigwedge^3 F \otimes A_1$$

restricted to the space of 6×6 traceless matrices $\ker(F \otimes F^* \to \mathbb{C})$ identified with $\mathbb{S}_{(2,1^4)} F^*$. The Betti table for the resolution is

	0	1	2	3	4	5
total:	1	20	35	35	20	1
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	.	20	35	.	.	.
3:	.	.	.	35	20	.
4:	.	.	.	.	.	.
5:	.	.	.	.	.	1

6.2.3 The orbit $\mathcal{O}_1$

The variety $\overline{\mathcal{O}}_1$ is the closure of the highest weight vector orbit. Geometrically, it is the cone over the Grassmannian $\mathrm{Gr}(3, \mathbb{C}^6)$. The defining ideal is generated by Plücker relations and can be obtained in M2 using `Grassmannian(2, 5, CoefficientRing => QQ)`. The Betti table of the minimal free resolution is

	0	1	2	3	4	5	6	7	8	9	10
total:	1	35	140	301	735	1080	735	301	140	35	1
0:	1	.	.	.	.	.	.	.	.	.	.
1:	.	35	140	189	.	.	.	.	.	.	.
2:	.	.	.	112	735	1080	735	112	.	.	.
3:	.	.	.	.	.	.	.	189	140	35	.
4:	.	.	.	.	.	.	.	.	.	.	1

This resolution was first determined by Pragacz and Weyman in [16].

6.3 The case (E_6, α_3)

The representation is $E \otimes \bigwedge^2 F$, where $E = \mathbb{C}^2$ and $F = \mathbb{C}^5$; the group acting is $\mathrm{SL}(E) \times \mathrm{SL}(F) \times \mathbb{C}^\times$. We denote the tensor $e_a \otimes f_i \wedge f_j$ by $[a; ij]$ where $a = 1, 2$ and $1 \leqslant i < j \leqslant 5$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{a;ij} | a = 1, 2; 1 \leqslant i < j \leqslant 5] = \mathrm{Sym}\left(E^* \otimes \bigwedge^2 F^*\right).$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	8	$x_{1;12} = 1$
$\mathcal{O}_2$	11	$x_{1;12} = x_{1;34} = 1$
$\mathcal{O}_3$	12	$x_{1;12} = x_{2;13} = 1$
$\mathcal{O}_4$	15	$x_{1;12} = x_{1;34} = x_{2;13} = 1$
$\mathcal{O}_5$	16	$x_{1;12} = x_{2;34} = 1$
$\mathcal{O}_6$	18	$x_{1;12} = x_{2;34} = x_{1;35} = 1$
$\mathcal{O}_7$	20	$x_{1;12} = x_{2;34} = x_{1;35} = x_{2;15} = 1$

All the orbit closures, except for $\overline{\mathcal{O}}_6$, are normal, Cohen-Macaulay and have rational singularities. Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$	$\overline{\mathcal{O}}_3$	$\overline{\mathcal{O}}_4$	$\overline{\mathcal{O}}_5$	$\overline{\mathcal{O}}_6$	$\overline{\mathcal{O}}_7$
$\mathcal{O}_0$	ns	s	s	s	s	s	s	ns
$\mathcal{O}_1$		ns	ns	s	s	s	s	ns
$\mathcal{O}_2$			ns		s	ns	s	ns
$\mathcal{O}_3$				ns	s	s	s	ns
$\mathcal{O}_4$					ns	ns	s	ns
$\mathcal{O}_5$						ns	s	ns
$\mathcal{O}_6$							ns	ns
$\mathcal{O}_7$								ns

We denote the free A-module $\mathbb{S}_{(a,b)} E^* \otimes \mathbb{S}_{(c,d,e,f,g)} F^* \otimes A(-a-b)$ by $(a, b; c, d, e, f, g)$.

6.3.1 The orbit $\mathcal{O}_6$

The orbit closure $\overline{\mathcal{O}}_6$ is not normal. The expected resolution for the coordinate ring of the normalization of $\overline{\mathcal{O}}_6$ is

$$A \oplus (1,1;1,1,1,1,0) \leftarrow (2,1;2,1,1,1,1) \leftarrow (4,1;2,2,2,2,2) \leftarrow 0$$

The differential $d_2 : (4,1;2,2,2,2,2) \to (2,1;2,1,1,1,1)$ was written explicitly as the map

$$\mathbb{S}_3 E^* \xrightarrow{\Delta} \mathbb{S}_2 E^* \otimes E^* \xrightarrow{tr^{(4)}} \mathbb{S}_2 E^* \otimes E^* \otimes \bigwedge^4 F \otimes \bigwedge^4 F^*$$

then $\mathbb{S}_2 E^* \otimes \bigwedge^4 F^*$ is embedded in A_2 via the map:

$$\begin{array}{c} \mathbb{S}_2 E^* \otimes \bigwedge^4 F^* \\ \downarrow \Delta \\ \mathbb{S}_2 E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\ \downarrow m_{2,3} \\ \mathbb{S}_2 E^* \otimes \mathbb{S}_2(\bigwedge^2 F^*) \\ \hookdownarrow \\ A_2 \end{array}$$

to get a map $\mathbb{S}_3 E^* \to E^* \otimes \bigwedge^4 F \otimes A_2$. The Betti table for the normalization is

	0	1	2
total:	6	10	4
0:	1	.	.
1:	.	.	.
2:	5	10	.
3:	.	.	4

Dropping the row of degree 3 in the first differential we get the map

$$(2,1;2,1,1,1,1) \to (1,1;1,1,1,1,0)$$

which is a presentation for the cokernel $C(6)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_6] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$. The resolution of $C(6)$ has Betti table

	0	1	2	3	4
total:	5	10	14	10	1
2:	5	10	.	.	.
3:	.	.	4	.	.
4:	.	.	10	10	.
5:	.	.	.	.	.
6:	.	.	.	.	1

We recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_6]$ using the cone procedure. Its equivariant form is

$$A \leftarrow (3,3;3,3,2,2,2) \leftarrow (4,3;3,3,3,3,2) \leftarrow (5,5;4,4,4,4,4) \leftarrow 0$$

and it has the following Betti table

	0	1	2	3
total:	1	10	10	1
0:	1	.	.	.
1:	.	.	.	.
2:	.	.	.	.
3:	.	.	.	.
4:	.	.	.	.
5:	.	10	10	.
6:	.	.	.	.
7:	.	.	.	1

We observe that $\overline{\mathcal{O}}_6$ is not Cohen-Macaulay because it has codimension 2 but its coordinate ring has projective dimension 3.

6.3.2 The orbit $\mathcal{O}_5$

The orbit closure $\overline{\mathcal{O}}_5$ is degenerate. The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_5]$ is

$$A \leftarrow (2,1;2,1,1,1,1) \leftarrow (4,1;2,2,2,2,2) \oplus (2,2;2,2,2,1,1) \leftarrow$$
$$\leftarrow (4,3;3,3,3,3,2) \leftarrow (4,4;4,3,3,3,3) \leftarrow 0$$

We construct explicitly the differential d_4 as follows

$$\begin{array}{c} F^* \\ \downarrow {\scriptstyle tr^{(1)} \otimes tr^{(1)}} \\ F^* \otimes E \otimes E^* \otimes F \otimes F^* \\ \downarrow {\scriptstyle m_{1,5}} \\ E \otimes E^* \otimes F \otimes \bigwedge^2 F^* \\ \downarrow {\scriptstyle m_{2,4}} \\ E \otimes F \otimes A_1 \end{array}$$

The Betti table for the resolution is

	0	1	2	3	4
total:	1	10	14	10	5
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	10	10	.	.
3:	.	.	4	.	.
4:	.	.	.	10	5

6.3.3 The orbit $\mathcal{O}_4$

The orbit closure $\overline{\mathcal{O}}_4$ is degenerate and comes from a smaller orbit which is a hypersurface of degree 4 in $E \otimes \bigwedge^2(\mathbb{C}^4) \hookrightarrow E \otimes \bigwedge^2 F$. This hypersurface is defined by the discriminant of the Pfaffian of a 4×4 skew-symmetric matrix of generic linear forms in two variables. The equations of $\overline{\mathcal{O}}_4$ in $E \otimes \bigwedge^2 F$ are obtained by taking polarizations of this discriminant with respect to the inclusion $E \otimes \bigwedge^2 \mathbb{C}^4 \hookrightarrow E \otimes \bigwedge^2 F$ (corresponding to the representation $\mathbb{S}_{(2,2)} E^* \otimes \mathbb{S}_{(2,2,2,2)} F^*$) together with the defining equations of the "generic degenerate orbit" $\overline{\mathcal{O}}_5$. The Betti table for the resolution is

	0	1	2	3	4	5
total:	1	25	62	55	20	3
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	.	10	10	.	.	.
3:	.	15	52	45	.	.
4:	.	.	.	10	20	.
5:	.	.	.	.	.	3

6.3.4 The orbit $\mathcal{O}_3$

The orbit closure $\overline{\mathcal{O}}_3$ is degenerate. The equations can be obtained directly through the map

$$\mathbb{S}_2\, E^* \otimes \bigwedge^4 F^* \hookrightarrow A_2$$

which is the embedding described in 6.3.1. The Betti table for the resolution is

	0	1	2	3	4	5	6	7	8
total:	1	15	75	187	265	245	121	20	5
0:	1	.	.	.	.	.	.	.	.
1:	.	15	20	.	.	.	.	.	.
2:	.	.	55	152	105	.	.	.	.
3:	.	.	.	35	160	245	120	.	.
4:	.	.	.	.	.	.	1	20	5

6.3.5 The orbit $\mathcal{O}_2$

The orbit closure $\overline{\mathcal{O}}_2$ is degenerate. The equations can be obtained directly through the map

$$\bigwedge^2 E^* \otimes \bigwedge^2 \left(\bigwedge^2 F^*\right) \hookrightarrow A_2$$

which gives the 2×2 minors of the generic matrix of a linear map $E^* \to \bigwedge^2 F^*$. The ideal is resolved by the Eagon-Northcott complex with Betti table

	0	1	2	3	4	5	6	7	8	9
total:	1	45	240	630	1008	1050	720	315	80	9
0:	1	.	.	.	.	.	.	.	.	.
1:	.	45	240	630	1008	1050	720	315	80	9

6.3.6 The orbit $\mathcal{O}_1$

The orbit closure $\overline{\mathcal{O}}_1$ is degenerate. The equations are simply those of the orbit closures $\overline{\mathcal{O}}_2$ and $\overline{\mathcal{O}}_3$ taken together. The Betti table of the resolution is

	0	1	2	3	4	5	6	7	8	9	10	11	12
total:	1	60	360	1011	1958	3750	5490	5235	3257	1329	375	60	4
0:	1	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	60	360	1005	1458	1050	720	315	80	9	.	.	.
2:	.	.	.	6	500	2700	4770	4920	3177	1200	285	30	.
3:	.	.	.	.	.	.	.	.	.	120	90	30	4

6.4 The case (E_6, α_4)

The representation is $E \otimes F \otimes H$, where $E = \mathbb{C}^2$ and $F = H = \mathbb{C}^3$; the group acting is $\mathrm{SL}(E) \times \mathrm{SL}(F) \times \mathrm{SL}(H) \times \mathbb{C}^\times$. We denote the tensor $e_i \otimes f_j \otimes h_k$ by $[i;j;k]$, where $i = 1,2$ and $j,k = 1,2,3$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{ijk} | i = 1,2; j,k = 1,2,3] = \mathrm{Sym}\,(E^* \otimes F^* \otimes H^*)$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative.

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	6	$x_{111} = 1$
$\mathcal{O}_2$	8	$x_{111} = x_{221} = 1$
$\mathcal{O}_3$	8	$x_{111} = x_{212} = 1$
$\mathcal{O}_4$	9	$x_{111} = x_{122} = 1$
$\mathcal{O}_5$	11	$x_{111} = x_{122} = x_{212} = 1$
$\mathcal{O}_6$	10	$x_{111} = x_{122} = x_{133} = 1$
$\mathcal{O}_7$	12	$x_{111} = x_{222} = 1$
$\mathcal{O}_8$	13	$x_{111} = x_{222} = x_{132} = 1$
$\mathcal{O}_9$	13	$x_{111} = x_{222} = x_{123} = 1$
$\mathcal{O}_{10}$	14	$x_{111} = x_{222} = x_{132} = x_{231} = 1$
$\mathcal{O}_{11}$	14	$x_{111} = x_{222} = x_{123} = x_{213} = 1$
$\mathcal{O}_{12}$	14	$x_{111} = x_{222} = x_{123} = x_{132} = 1$
$\mathcal{O}_{13}$	14	$x_{111} = x_{222} = x_{123} = x_{231} = 1$
$\mathcal{O}_{14}$	15	$x_{111} = x_{222} = x_{133} = 1$
$\mathcal{O}_{15}$	16	$x_{111} = x_{222} = x_{123} = x_{231} = x_{132} = 1$
$\mathcal{O}_{16}$	17	$x_{111} = x_{222} = x_{133} = x_{213} = 1$
$\mathcal{O}_{17}$	18	$x_{111} = x_{211} = x_{122} = x_{133} = 1, x_{222} = -1$

The containment and singularity table can be found below.

We denote the free A-module $\mathbb{S}_{(a,b)}\,E^* \otimes \mathbb{S}_{(c,d,e)}\,F^* \otimes \mathbb{S}_{(f,g,h)}\,F^* \otimes A(-a-b)$ by $(a,b;c,d,e;f,g,h)$.

Certain pairs of orbit closures are isomorphic under the involution exchanging F and H. This involution produces an automorphism of A exchanging x_{ijk} with x_{ikj}; this, in turn, induces isomorphisms of the coordinate rings and free resolutions. Because of this, it is enough to discuss only one case in each pair.

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$	$\overline{\mathcal{O}}_3$	$\overline{\mathcal{O}}_4$	$\overline{\mathcal{O}}_5$	$\overline{\mathcal{O}}_6$	$\overline{\mathcal{O}}_7$	$\overline{\mathcal{O}}_8$	$\overline{\mathcal{O}}_9$	$\overline{\mathcal{O}}_{10}$	$\overline{\mathcal{O}}_{11}$	$\overline{\mathcal{O}}_{12}$	$\overline{\mathcal{O}}_{13}$	$\overline{\mathcal{O}}_{14}$	$\overline{\mathcal{O}}_{15}$	$\overline{\mathcal{O}}_{16}$	$\overline{\mathcal{O}}_{17}$
$\mathcal{O}_0$	ns	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	s	ns
$\mathcal{O}_1$		ns	ns	ns	s	s	ns	s	s	s	s	s	s	s	s	s	s	ns
$\mathcal{O}_2$			ns			s		s	s	s	s	ns	s	s	s	s	s	ns
$\mathcal{O}_3$				ns		s		s	s	s	ns	s	s	s	s	s	s	ns
$\mathcal{O}_4$					ns	s	ns	ns	s	s	ns	ns	s	s	s	s	s	ns
$\mathcal{O}_5$						ns		ns	s	s	ns	ns	s	s	s	s	s	ns
$\mathcal{O}_6$							ns						s		s	s	s	ns
$\mathcal{O}_7$								ns	s	s	ns	ns	s	s	s	s	s	ns
$\mathcal{O}_8$									ns		ns		ns	ns	ns	s	s	ns
$\mathcal{O}_9$										ns		ns	ns	ns	ns	s	s	ns
$\mathcal{O}_{10}$											ns					s	s	ns
$\mathcal{O}_{11}$												ns				s	s	ns
$\mathcal{O}_{12}$													ns		ns	s	s	ns
$\mathcal{O}_{13}$														ns		s	s	ns
$\mathcal{O}_{14}$															ns		s	ns
$\mathcal{O}_{15}$																ns	s	ns
$\mathcal{O}_{16}$																	ns	ns
$\mathcal{O}_{17}$																		ns

Table 6.1: Containment and singularity table for the case (E_6, α_4)

6.4.1 The orbit $\mathcal{O}_{16}$

The orbit closure $\overline{\mathcal{O}}_{16}$ is a hypersurface defined by the discriminant of the determinant of a generic 3×3 matrix of linear forms in two variables, which is a homogeneous polynomial of degree 12. Explicitly:

$$\begin{aligned}\delta = \det &\begin{pmatrix} ux_{111} + vx_{211} & ux_{112} + vx_{212} & ux_{113} + vx_{213} \\ ux_{121} + vx_{221} & ux_{122} + vx_{222} & ux_{123} + vx_{223} \\ ux_{131} + vx_{231} & ux_{132} + vx_{232} & ux_{133} + vx_{233} \end{pmatrix} = \\ &= a_{3,0}u^3 + a_{2,1}u^2v + a_{1,2}uv^2 + a_{0,3}v^3,\end{aligned}$$

and

$$\operatorname{disc}(\delta) = 27a_{3,0}^2a_{1,2}^2 + 4a_{3,0}a_{1,2}^3 + 4a_{2,1}^3a_{0,3} - a_{2,1}^2a_{1,2}^2 - 18a_{3,0}a_{2,1}a_{1,2}a_{0,3}.$$

The orbit closure $\overline{\mathcal{O}}_{16}$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_{16})$ is

$$A \oplus (2,1;1,1,1;1,1,1) \leftarrow (4,2;2,2,2;2,2,2) \leftarrow 0$$

so there is only one differential d_1 with two blocks. The block $(4,2;2,2,2;2,2,2) \to (2,1;1,1,1;1,1,1)$ is the map

$$\begin{array}{c} \mathbb{S}_2 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \\ \big\downarrow {\scriptstyle tr^{(1)}} \\ E \otimes E^* \otimes \mathbb{S}_2 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \\ \big\downarrow {\scriptstyle m_{2,3}} \\ E \otimes \mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \\ \big\downarrow \\ E \otimes A_3 \end{array}$$

where the embedding $\mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \hookrightarrow A_3$ at the end is given by

$$(e_1^*)^i (e_2^*)^j \longmapsto \frac{i!j!}{3!} a_{i,j},$$

the $a_{i,j}$ being the coefficients of δ as above. Notice that this block also provides a minimal presentation of the cokernel $C(16)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{16}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{16})]$. The block $(4,2;2,2,2;2,2,2) \to A$ can be obtained as follows. First construct the following map for the E^* factor:

$$\begin{array}{c} \mathbb{S}_2 E^* \otimes \bigwedge^2 E^* \otimes \bigwedge^2 E^* \\ \Big\downarrow {\scriptstyle \Delta \otimes \Delta \otimes \Delta} \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \\ \Big\downarrow {\scriptstyle m_{1,3,5} \otimes m_{2,4,6}} \\ \mathbb{S}_3 E^* \otimes \mathbb{S}_3 E^* \end{array}$$

Then embed into A via the map

$$\begin{array}{c} \mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \otimes \mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \\ \Big\downarrow \\ A_3 \otimes A_3 \\ \Big\downarrow \\ A_6 \end{array}$$

where the first step uses the embedding described earlier twice and the second step is symmetric multiplication.

The Betti table for the resolution of the normalization is

	0	1
total:	3	3
0:	1	.
1:	.	.
2:	.	.
3:	2	.
4:	.	.
5:	.	3

and the Betti table for the resolution of the cokernel $C(16)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{16}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{16})]$ is

	0	1	2
total:	2	3	1
3:	2	.	.
4:	.	.	.
5:	.	3	.
6:	.	.	.
7:	.	.	.
8:	.	.	.
9:	.	.	.
10:	.	.	1

6.4.2 The orbit $\mathcal{O}_{15}$

The orbit closure $\overline{\mathcal{O}}_{15}$ is normal with rational singularities. The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_{15}]$ is

$$A \longleftarrow (4,2;2,2,2;2,2,2) \longleftarrow (5,4;3,3,3;3,3,3) \longleftarrow 0$$

The differential d_2 was written explicitly by taking the map

$$\begin{array}{c} \bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes E^* \\ \Big\downarrow \Delta\otimes\Delta \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \\ \Big\downarrow m_{1,3}\otimes m_{2,4} \\ \mathbb{S}_2 E^* \otimes \mathbb{S}_2 E^* \otimes E^* \\ \Big\downarrow m_{1,3} \\ \mathbb{S}_3 E^* \otimes \mathbb{S}_2 E^* \end{array}$$

on the E^* factor and then embedding $\mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^*$ into A_3 as described in 6.4.1. The Betti table for the resolution is

	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	.	.
2:	.	.	.
3:	.	.	.
4:	.	.	.
5:	.	3	.
6:	.	.	.
7:	.	.	2

We conclude that $\overline{\mathcal{O}}_{15}$ is Cohen-Macaulay.

6.4.3 The orbit $\mathcal{O}_{14}$

The orbit closure $\overline{\mathcal{O}}_{14}$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_{14})$ is

$$A \oplus (1,1;1,1,0;1,1,0) \leftarrow (2,1;1,1,1;2,1,0) \oplus (2,1;2,1,0;1,1,1) \leftarrow$$
$$\leftarrow (3,1;2,1,1;2,1,1) \leftarrow (5,1;2,2,2;2,2,2) \leftarrow 0$$

The differential $d_3 : (5,1;2,2,2;2,2,2) \to (3,1;2,1,1;2,1,1)$ was written explicitly as the map

$$\begin{array}{c} \mathbb{S}_4 E^* \\ \Big\downarrow {\scriptstyle \Delta \otimes tr^{(2)} \otimes tr^{(2)}} \\ \mathbb{S}_2 E^* \otimes \mathbb{S}_2 E^* \otimes \bigwedge^2 F \otimes \bigwedge^2 F^* \otimes \bigwedge^2 H \otimes \bigwedge^2 H^* \\ \Big\downarrow \\ \mathbb{S}_2 E^* \otimes \bigwedge^2 F \otimes \bigwedge^2 H \otimes A_2 \end{array}$$

The embedding $\mathbb{S}_2 E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 H^* \hookrightarrow A_2$ works by sending the basis vector

$$(e_1^*)^i (e_2^*)^j \otimes f_a \wedge f_b \otimes h_c \wedge h_d$$

to the coefficient of $u^i v^j$ in the expansion of

$$\frac{i!j!}{2!}\det\begin{pmatrix} ux_{1ac}+vx_{2ac} & ux_{1ad}+vx_{2ad} \\ ux_{1bc}+vx_{2bc} & ux_{1bd}+vx_{2bd} \end{pmatrix}$$

Notice that the minor above is the one corresponding to rows a, b and columns c, d in the matrix δ defined in 6.4.1. The Betti table for the normalization is

	0	1	2	3
total:	10	32	27	5
0:	1	.	.	.
1:	.	.	.	.
2:	9	32	27	.
3:	.	.	.	5

Dropping the row of degree 3 in the first differential we get the map

$$(2,1;1,1,1;2,1,0)\oplus(2,1;2,1,0;1,1,1)\to(1,1;1,1,0;1,1,0)$$

This is a presentation for the cokernel $C(14)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{14}]\hookrightarrow\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{14})]$; the Betti table for $C(14)$ is

	0	1	2	3	4	5	6	7	8
total:	9	32	131	347	477	372	181	54	7
2:	9	32	27	.	.	.	.	.	.
3:	.	.	.	5	.	.	.	.	.
4:	.	.	104	342	477	372	180	54	7
5:	.	.	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	1	.	.

By the truncated cone procedure, we recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_{14}]$. Its equivariant form is

$$A \leftarrow (3,3;2,2,2;3,3,0) \oplus (3,3;3,3,0;2,2,2) \oplus (3,3;3,2,1;3,2,1) \oplus$$
$$\oplus (3,3;2,2,2;4,1,1) \oplus (3,3;4,1,1;2,2,2) \leftarrow$$
$$\leftarrow (4,3;3,3,1;3,3,1) \oplus (4,3;3,3,1,3,2,2) \oplus (4,3;3,2,2;3,3,1) \oplus$$
$$\oplus (4,3;3,2,2;3,2,2) \oplus (4,3;3,2,2;4,2,1) \oplus (4,3;4,2,1;3,2,2) \leftarrow$$
$$\leftarrow (4,4;3,3,2;3,3,2) \oplus (4,4;3,3,2;4,2,2) \oplus (4,4;4,2,2;3,3,2) \oplus$$
$$\oplus 2 * (5,3;3,3,2;3,3,2) \oplus (5,3;3,3,2;4,3,1) \oplus$$
$$\oplus (5,3;4,3,1;3,3,2) \oplus (5,3;4,2,2;4,2,2) \leftarrow$$
$$\leftarrow (5,4;3,3,3;4,3,2) \oplus (5,4;4,3,2;3,3,3) \oplus (6,3;3,3,3;3,3,3) \oplus$$
$$\oplus (6,3;3,3,3;4,4,1) \oplus (6,3;4,4,1;3,3,3) \oplus (6,3;4,3,2;4,3,2) \leftarrow$$
$$\leftarrow (6,6;4,4,4;4,4,4) \oplus (7,3;4,4,2;4,3,3) \oplus (7,3;4,3,3;4,4,2) \leftarrow$$
$$\leftarrow (8,3;4,4,3;4,4,3) \leftarrow (9,3;4,4,4;4,4,4) \leftarrow 0$$

and it has the following Betti table

	0	1	2	3	4	5	6	7
total:	1	104	342	477	372	181	54	7
0:	1	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	.	.
3:	.	.	.	.	.	.	.	.
4:	.	.	.	.	.	.	.	.
5:	.	104	342	477	372	180	54	7
6:	.	.	.	.	.	.	.	.
7:	.	.	.	.	.	1	.	.

We observe that $\overline{\mathcal{O}}_{14}$ is not Cohen-Macaulay because it has codimension 3 but its coordinate ring has projective dimension 7.

6.4.4 The orbit $\mathcal{O}_{13}$

The orbit closure $\overline{\mathcal{O}}_{13}$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_{13})$ is

$$A \oplus (1,1;1,1,0;1,1,0) \leftarrow (2,1;1,1,1;1,1,1)\oplus$$
$$\oplus(2,1;1,1,1;2,1,0) \oplus (2,1;2,1,0;1,1,1) \oplus (3,0;1,1,1;1,1,1) \leftarrow$$
$$\leftarrow (2,2;2,1,1;2,1,1) \oplus (3,1;2,1,1;2,1,1) \oplus (3,3;2,2,2;2,2,2) \leftarrow$$
$$\leftarrow (4,3;3,2,2;3,2,2) \leftarrow (4,4;3,3,2;3,3,2) \leftarrow 0$$

The differential $d_4 : (4,4;3,3,2;3,3,2) \to (4,3;3,2,2;3,2,2)$ was written explicitly as the map

$$\begin{array}{c} \bigwedge^2 F^* \otimes \bigwedge^2 H^* \\ \Big\downarrow {\scriptstyle tr^{(1)} \otimes \Delta \otimes \Delta} \\ E \otimes E^* \otimes F^* \otimes F^* \otimes H^* \otimes H^* \\ \Big\downarrow {\scriptstyle m_{2,4,6}} \\ E \otimes F^* \otimes H^* \otimes A_1 \end{array}$$

The Betti table for the normalization is

	0	1	2	3	4
total:	10	38	37	18	9
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	9	38	36	.	.
3:	.	.	.	.	.
4:	.	.	1	18	9

Dropping the row of degree 3 in the first differential we get the map

$$(2,1;1,1,1;1,1,1) \oplus (2,1;1,1,1;2,1,0) \oplus (2,1;2,1,0;1,1,1) \to (1,1;1,1,0;1,1,0)$$

Notice how the representation $(3,0;1,1,1;1,1,1)$ was also dropped from the domain since it does not map to $(1,1;1,1,0;1,1,0)$. This is a presentation for the cokernel $C(13)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{13}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{13})]$, whose Betti table is

	0	1	2	3	4	5	6
total:	9	34	56	95	99	36	1
2:	9	34	36	.	.	.	.
3:	.	.	.	5	.	.	.
4:	.	.	20	90	99	36	.
5:	.	.	.	.	.	.	.
6:	.	.	.	.	.	.	1

By the truncated cone procedure, we recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_{13}]$. Its equivariant form is

$$A \leftarrow (3,0;1,1,1;1,1,1) \oplus (3,3;2,2,2;3,3,0) \oplus (3,3;3,3,0;2,2,2) \leftarrow$$
$$\leftarrow (3,3;2,2,2;2,2,2) \oplus (5,1;2,2,2;2,2,2) \oplus (4,3;3,3,1;3,2,2) \oplus (4,3;3,2,2;3,3,1) \leftarrow$$
$$\leftarrow (4,4;3,3,2;4,2,2) \oplus (4,4;4,2,2;3,3,2) \oplus 2*(5,3;3,3,2;3,3,2) \leftarrow$$
$$\leftarrow (5,4;3,3,3;4,3,2) \oplus (5,4;4,3,2;3,3,3) \oplus (6,3;3,3,3;3,3,3) \leftarrow (6,6;4,4,4;4,4,4) \leftarrow 0$$

which has the following Betti table

	0	1	2	3	4	5
total:	1	24	78	90	36	1
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	.	4	.	.	.	.
3:	.	.	.	.	.	.
4:	.	.	6	.	.	.
5:	.	20	72	90	36	.
6:	.	.	.	.	.	.
7:	.	.	.	.	.	1

We observe that $\overline{\mathcal{O}}_{13}$ is not Cohen-Macaulay because it has codimension 4 but its coordinate ring has projective dimension 5.

6.4.5 The orbit $\mathcal{O}_{12}$

The orbit closure $\overline{\mathcal{O}}_{12}$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_{12})$ is

$$A \oplus (1,1;1,1,0;1,1,0) \leftarrow$$
$$\leftarrow (2,1;1,1,1;1,1,1) \oplus (2,1;1,1,1;2,1,0) \oplus (2,1;2,1,0;1,1,1) \leftarrow$$
$$\leftarrow (3,1;2,1,1;2,1,1) \oplus (3,3;2,2,2;3,2,1) \oplus (3,3;3,2,1;2,2,2) \leftarrow$$
$$\leftarrow (4,3;3,2,2;3,2,2) \oplus (5,1;2,2,2;2,2,2) \leftarrow (6,3;3,3,3;3,3,3) \leftarrow 0$$

The differential $d_4 : (6,3;3,3,3;3,3,3) \to (4,3;3,2,2;3,2,2) \oplus (5,1;2,2,2;2,2,2)$ was written explicitly. The block $(6,3;3,3,3;3,3,3) \to (4,3;3,2,2;3,2,2)$ was constructed as the map

$$\begin{array}{c} \mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^* \\ \Big\downarrow \Delta\otimes\Delta\otimes\Delta \\ E^* \otimes \mathbb{S}_2 E^* \otimes F^* \otimes \bigwedge^2 F^* \otimes H^* \otimes \bigwedge^2 H^* \\ \Big\downarrow m_{2,4,6} \\ E^* \otimes F^* \otimes H^* \otimes A_2 \end{array}$$

where the embedding $\mathbb{S}_2 E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 H^* \hookrightarrow A_2$ is the one described in 6.4.3. The second block corresponding to the map $(6,3;3,3,3;3,3,3) \to (5,1;2,2,2;2,2,2)$ was constructed as the map

$$\begin{array}{c} \bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes \mathbb{S}_3 E^* \\ \Big\downarrow \Delta\otimes\Delta\otimes\Delta \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes \mathbb{S}_2 E^* \otimes E^* \\ \Big\downarrow m_{1,3,5}\otimes m_{2,4,6} \\ \mathbb{S}_4 E^* \otimes \mathbb{S}_3 E^* \end{array}$$

on the E^* factor and then embedding $\mathbb{S}_3 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 H^*$ into A_3 as described in 6.4.1. The Betti table for the normalization is

	0	1	2	3	4
total:	10	34	43	23	4
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	9	34	27	.	.
3:	.	.	.	5	.
4:	.	.	16	18	.
5:	.	.	.	.	4

Dropping the row of degree 3 in the first differential we get the map

$$(2,1;1,1,1;1,1,1) \oplus (2,1;1,1,1;2,1,0) \oplus (2,1;2,1,0;1,1,1) \to (1,1;1,1,0;1,1,0)$$

which is a presentation for the cokernel $C(12)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{12}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{12})]$. Notice that this is the same as the presentation of $C(13)$ whose Betti table is described in 6.4.4. By the truncated cone procedure, we recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_{12}]$. Its equivariant form is

$$A \leftarrow (2,2;2,1,1;2,1,1) \oplus (3,3;2,2,2;3,3,0) \oplus (3,3;3,3,0;2,2,2) \leftarrow$$
$$\leftarrow (3,3;2,2,2;3,2,1) \oplus (3,3;3,2,1;2,2,2) \oplus (4,3;3,3,1;3,2,2) \oplus (4,3;3,2,2;3,3,1) \leftarrow$$
$$\leftarrow (4,4;3,3,2;3,3,2) \oplus (4,4;3,3,2;4,2,2) \oplus (4,4;4,2,2;3,3,2) \oplus 2*(5,3;3,3,2;3,3,2) \leftarrow$$
$$\leftarrow (5,4;3,3,3;4,3,2) \oplus (5,4;4,3,2;3,3,3) \oplus 2*(6,3;3,3,3;3,3,3) \leftarrow$$
$$\leftarrow (6,6;4,4,4;4,4,4) \leftarrow 0$$

and it has the following Betti table

	0	1	2	3	4	5
total:	1	29	88	99	40	1
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	.	.	.	.	.	.
3:	.	9	.	.	.	.
4:	.	.	16	.	.	.
5:	.	20	72	99	40	.
6:	.	.	.	.	.	.
7:	.	.	.	.	.	1

We observe that $\overline{\mathcal{O}}_{12}$ is not Cohen-Macaulay because it has codimension 4 but its coordinate ring has projective dimension 5.

6.4.6 The orbits $\mathcal{O}_{11}$ and $\mathcal{O}_{10}$

We discuss the case of the orbit $\mathcal{O}_{11}$ since $\mathcal{O}_{10}$ is isomorphic under the involution exchanging F and H. The orbit closure $\overline{\mathcal{O}}_{11}$ is normal with rational singularities. It is F-degenerate and its equations are the 3×3 minors of

$$\begin{pmatrix} x_{111} & x_{121} & x_{131} \\ x_{112} & x_{122} & x_{132} \\ x_{113} & x_{123} & x_{133} \\ x_{211} & x_{221} & x_{231} \\ x_{212} & x_{222} & x_{232} \\ x_{213} & x_{223} & x_{233} \end{pmatrix}$$

the generic matrix of a linear map $F \to E^* \otimes H^*$. As such $\overline{\mathcal{O}}_{11}$ is a determinantal variety and its resolution is given by the Eagon-Northcott complex with the following Betti table

	0	1	2	3	4
total:	1	20	45	36	10
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	20	45	36	10

It follows that $\overline{\mathcal{O}}_{11}$ is Cohen-Macaulay.

6.4.7 The orbits $\mathcal{O}_{9}$ and $\mathcal{O}_{8}$

We discuss the case of the orbit $\mathcal{O}_{8}$ since $\mathcal{O}_{9}$ is isomorphic under the involution exchanging F and H. The orbit closure $\overline{\mathcal{O}}_{8}$ is normal with rational singularities. It is degenerate and comes from a smaller orbit which is a hypersurface of degree 6 in the representation

$$\mathbb{S}_{(3,3)}\, E^* \otimes \mathbb{S}_{(2,2,2)}\, F^* \otimes \mathbb{S}_{(3,3)}(\mathbb{C}^2)^* \hookrightarrow \mathbb{S}_{(3,3)}\, E^* \otimes \mathbb{S}_{(2,2,2)}\, F^* \otimes \mathbb{S}_{(3,3)}\, H^*.$$

The hypersurface is defined by the invariant of degree 6 in $\mathbb{C}^2 \otimes \mathbb{C}^3 \otimes \mathbb{C}^2$ which is the hyperdeterminant of the boundary format $2 \times 3 \times 2$ (see [4]). The equations of $\overline{\mathcal{O}}_8$ are obtained by taking polarizations of such invariant with respect to the inclusion above together with the equations of the "generic degenerate orbit" $\overline{\mathcal{O}}_{10}$. The Betti table for the resolution is

	0	1	2	3	4	5
total:	1	30	81	81	30	1
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	.	20	45	36	10	.
3:	.	.	.	.	.	.
4:	.	.	.	.	.	.
5:	.	10	36	45	20	.
6:	.	.	.	.	.	.
7:	.	.	.	.	.	1

It follows that $\overline{\mathcal{O}}_8$ is Gorenstein.

6.4.8 The orbit $\mathcal{O}_7$

The orbit closure $\overline{\mathcal{O}}_7$ is normal with rational singularities. It is F-H-degenerate with equations given by the 3×3 minors of the generic matrix of a linear map $F \to E^* \otimes H^*$ together with the 3×3 minors of the generic matrix of a linear map $H \to E^* \otimes F^*$; in other words, these are the equations of $\overline{\mathcal{O}}_{11}$ and $\overline{\mathcal{O}}_{10}$ taken together. The Betti table for the resolution is

	0	1	2	3	4	5	6
total:	1	36	99	95	56	34	9
0:	1	.	.	.	.	.	.
1:	.	.	.	.	.	.	.
2:	.	36	99	90	20	.	.
3:	.	.	.	5	.	.	.
4:	.	.	.	.	36	34	9

It follows that $\overline{\mathcal{O}}_7$ is Cohen-Macaulay.

6.4.9 The orbit $\mathcal{O}_6$

The orbit closure $\overline{\mathcal{O}}_6$ is normal with rational singularities. It is E-degenerate and its equations are the 2×2 minors of

$$\begin{pmatrix} x_{111} & x_{211} \\ x_{112} & x_{212} \\ x_{113} & x_{213} \\ x_{121} & x_{221} \\ x_{122} & x_{222} \\ x_{123} & x_{223} \\ x_{131} & x_{231} \\ x_{132} & x_{232} \\ x_{133} & x_{233} \end{pmatrix}$$

the generic matrix of a linear map $E \to F^* \otimes H^*$. Therefore $\overline{\mathcal{O}}_6$ is a determinantal variety and its resolution is given by the Eagon-Northcott complex with the following Betti table

	0	1	2	3	4	5	6	7	8
total:	1	36	168	378	504	420	216	63	8
0:	1	.	.	.	.	.	.	.	.
1:	.	36	168	378	504	420	216	63	8

It follows that $\overline{\mathcal{O}}_6$ is Cohen-Macaulay.

6.4.10 The orbit $\mathcal{O}_5$

The orbit closure $\overline{\mathcal{O}}_5$ is normal with rational singularities. It is degenerate and comes from a smaller orbit which is a hypersurface of degree 4 in the representation

$$\mathbb{S}_{(2,2)} E^* \otimes \mathbb{S}_{(2,2)}(\mathbb{C}^2)^* \otimes \mathbb{S}_{(2,2)}(\mathbb{C}^2)^* \hookrightarrow \mathbb{S}_{(2,2)} E^* \otimes \mathbb{S}_{(2,2)} F^* \otimes \mathbb{S}_{(2,2)} H^*.$$

The hypersurface is defined by the invariant of degree 4 in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ which can be written as the discriminant of the determinant of a generic 2×2 matrix of linear forms in two variables.

Explicitly take:

$$\det\begin{pmatrix} ux_{111}+vx_{211} & ux_{112}+vx_{212} \\ ux_{121}+vx_{221} & ux_{122}+vx_{222}\end{pmatrix} = a_{2,0}u^2 + a_{1,1}uv + a_{0,2}v^2$$

and the discriminant is $4a_{2,0}a_{0,2} - a_{1,1}^2$. The equations of $\overline{\mathcal{O}}_5$ are obtained by taking polarizations of such invariant with respect to the inclusion above together with the equations of the "generic degenerate orbit" $\overline{\mathcal{O}}_7$. The Betti table for the resolution is

	0	1	2	3	4	5	6	7
total:	1	72	297	530	488	223	42	3
0:	1	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.
2:	.	36	99	90	20	.	.	.
3:	.	36	198	440	468	189	6	.
4:	.	.	.	.	.	34	36	.
5:	.	.	.	.	.	.	.	3

It follows that $\overline{\mathcal{O}}_5$ is Cohen-Macaulay.

6.4.11 The orbit $\mathcal{O}_4$

The orbit closure $\overline{\mathcal{O}}_4$ is normal with rational singularities. It is both E-degenerate and F-H-degenerate with equations given by the 2×2 minors of the generic matrix of a linear map $E \to F^* \otimes H^*$ together with the coefficients of the determinant of a generic 3×3 matrix of linear forms in two variables. More explicitly, the former are the equations of $\overline{\mathcal{O}}_6$ while the latter are the coefficients $a_{3,0}, a_{2,1}, a_{1,2}, a_{0,3}$ of $\det(\delta)$ as introduced in 6.4.1. The Betti table for the resolution is

	0	1	2	3	4	5	6	7	8	9
total:	1	40	195	450	588	546	384	171	44	5
0:	1	.	.	.	.	.	.	.	.	.
1:	.	36	168	378	504	420	216	63	8	.
2:	.	4	27	72	84	.	.	.	.	.
3:	.	.	.	.	.	126	168	108	36	5

It follows that $\overline{\mathcal{O}}_4$ is Cohen-Macaulay.

6.4.12 The orbits $\mathcal{O}_3$ and $\mathcal{O}_2$

We discuss the case of the orbit $\mathcal{O}_3$ since $\mathcal{O}_2$ is isomorphic under the involution exchanging F and H. The orbit closure $\overline{\mathcal{O}}_3$ is normal with rational singularities. It is F-degenerate and its equations are the 2×2 minors of the generic matrix of a linear map $F \to E^* \otimes H^*$ (see 6.4.6). As such $\overline{\mathcal{O}}_3$ is a determinantal variety and the coordinate ring is resolved by Lascoux's resolution [20]. Here is the Betti table for the resolution

	0	1	2	3	4	5	6	7	8	9	10
total:	1	45	230	540	823	1015	1035	760	351	90	10
0:	1	.	.	.	.	.	.	.	.	.	.
1:	.	45	230	540	648	385	90	.	.	.	.
2:	.	.	.	.	175	630	945	760	351	90	10

It follows that $\overline{\mathcal{O}}_3$ is Cohen-Macaulay.

6.4.13 The orbit $\mathcal{O}_1$

The orbit closure $\overline{\mathcal{O}}_1$ is normal with rational singularities. It is both E-degenerate and F-H-degenerate with equations given by the 2×2 minors of the generic matrix of a linear map $E \to F^* \otimes H^*$ together with the coefficients of the 2×2 minors of a generic 3×3 matrix of linear forms in two variables. More explicitly, the former are the equations of $\overline{\mathcal{O}}_6$ while the latter are the coefficients of the 2×2 minors of δ as introduced in 6.4.1. The Betti table for the resolution is

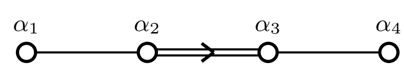
	0	1	2	3	4	5	6	7	8	9	10	11	12
total:	1	63	394	1179	2087	2692	3726	4383	3275	1530	407	45	2
0:	1	.	.	.	.	.	.	.	.	.	.	.	.
1:	.	63	394	1179	1980	1702	396	63	8	.	.	.	.
2:	.	.	.	.	107	990	3330	4320	3267	1530	407	36	.
3:	.	.	.	.	.	.	.	.	.	.	.	9	2

It follows that $\overline{\mathcal{O}}_1$ is Cohen-Macaulay.

Chapter 7

Representations of type F_4

In this chapter, we will analyze the cases corresponding to gradings on the simple Lie algebra of type F_4. Each case corresponds to the choice of a distinguished node on the Dynkin diagram for F_4. The nodes are numbered according to the conventions in Bourbaki [1].



7.1 The case (F_4, α_1)

The representation is $V = V(\omega_3, C_3)$, the third fundamental representation of the group $\mathrm{Sp}(6, \mathbb{C})$; the group acting is $\mathbb{C}^\times \times \mathrm{Sp}(6, \mathbb{C})$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{1342}, x_{1242}, x_{1232}, x_{1231}, x_{1222}, x_{1221}, x_{1122},$$
$$x_{1220}, x_{1121}, x_{1120}, x_{1111}, x_{1110}, x_{1100}, x_{1000}] = \mathrm{Sym}(V^*).$$

The variables in A are indexed by the roots in $\mathfrak{g}_1 = V$. The variables are weight vectors in V^*, with the following weights:

$$\begin{array}{ll} x_{1342} \leftrightarrow \epsilon_1 + \epsilon_2 + \epsilon_3 & x_{1242} \leftrightarrow \epsilon_1 + \epsilon_2 - \epsilon_3 \\ x_{1232} \leftrightarrow \epsilon_1 & x_{1231} \leftrightarrow \epsilon_2 \\ x_{1222} \leftrightarrow \epsilon_1 - \epsilon_2 + \epsilon_3 & x_{1221} \leftrightarrow \epsilon_3 \\ x_{1122} \leftrightarrow \epsilon_1 - \epsilon_2 - \epsilon_3 & x_{1220} \leftrightarrow -\epsilon_1 + \epsilon_2 + \epsilon_3 \\ x_{1121} \leftrightarrow -\epsilon_3 & x_{1120} \leftrightarrow -\epsilon_1 + \epsilon_2 - \epsilon_3 \\ x_{1111} \leftrightarrow -\epsilon_2 & x_{1110} \leftrightarrow -\epsilon_1 \\ x_{1100} \leftrightarrow -\epsilon_1 - \epsilon_2 + \epsilon_3 & x_{1000} \leftrightarrow -\epsilon_1 - \epsilon_2 - \epsilon_3 \end{array}$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	7	$x_{1000} = 1$
$\mathcal{O}_2$	10	$x_{1111} = 1$
$\mathcal{O}_3$	13	$x_{1000} = x_{1231} = 1$
$\mathcal{O}_4$	14	$x_{1000} = x_{1342} = 1$

All the orbit closures are normal, Cohen-Macaulay, and have rational singularities. Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$	$\overline{\mathcal{O}}_3$	$\overline{\mathcal{O}}_4$
$\mathcal{O}_0$	ns	s	s	s	ns
$\mathcal{O}_1$		ns	s	s	ns
$\mathcal{O}_2$			ns	s	ns
$\mathcal{O}_3$				ns	ns
$\mathcal{O}_4$					ns

We will denote by V_ω the highest weight module with highest weight ω. The weights will be expressed as linear combinations of ω_1, ω_2, ω_3, the fundamental weights of the root system C_3.

Let $F = V_{\omega_1} = \mathbb{C}^6$ be the standard representation of $\mathrm{Sp}(6, \mathbb{C})$. Since $\bigwedge^3 F \cong F \oplus V$, there is a projection $\phi : \bigwedge^3 F \to V$ of $\mathrm{Sp}(6, \mathbb{C})$-modules. Take $\psi : V \to \bigwedge^3 F$, to be a section of ϕ, so

$\phi \circ \psi = \mathrm{id}$. Also, let $\delta : F \to F^*$ be the duality given by the symplectic form on F.

Next we observe that $A_2 = \mathbb{S}_2 V^* \cong \mathbb{S}_2 F^* \oplus V^*_{2\omega_2}$. Therefore there exists a non zero map $\rho : \mathbb{S}_2 F^* \to A_2$ of $\mathrm{Sp}(6,\mathbb{C})$-modules. We explain here one possible way to write such a map explicitly, in terms of well understood equivariant maps:

$$\begin{array}{c} \mathbb{S}_2 F^* \\ \downarrow \Delta \\ F^* \otimes F^* \\ \downarrow \delta^{-1} \\ F \otimes F^* \\ \downarrow * \\ \bigwedge^5 F^* \otimes F^* \\ \downarrow \Delta \\ \bigwedge^3 F^* \otimes \bigwedge^2 F^* \otimes F^* \\ \downarrow m_{2,3} \\ \bigwedge^3 F^* \otimes \bigwedge^3 F^* \\ \downarrow \psi^* \otimes \psi^* \\ V^* \otimes V^* \\ \downarrow m_{1,2} \\ \mathbb{S}_2 V^* \end{array}$$

This map will be used in the description of the orbit closures for the case (F_4, α_1).

7.1.1 The orbit $\mathcal{O}_3$

The orbit closure $\overline{\mathcal{O}}_3$ is a hypersurface defined by an invariant of degree 4 which can be obtained as follows:

$$\mathbb{C} \xrightarrow{tr^{(2)}} \mathbb{S}_2 F \otimes \mathbb{S}_2 F^* \xrightarrow{\mathbb{S}_2(\delta)} \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \xrightarrow{\rho \otimes \rho} A_2 \otimes A_2 \longrightarrow A_4$$

where the last map is symmetric multiplication. Another description of this invariant was given by Landsberg and Manivel [12].

7.1.2 The orbit $\mathcal{O}_2$

The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_2]$ is

$$A \leftarrow V_{\omega_3} \otimes A(-3) \leftarrow V_{2\omega_1} \otimes A(-4) \leftarrow$$
$$\leftarrow V_{\omega_2} \otimes A(-6) \leftarrow V_{\omega_1} \otimes A(-7) \leftarrow 0$$

The differential $d_2 : V_{2\omega_1} \otimes A(-4) \to V_{\omega_3} \otimes A(-3)$ was written explicitly, as follows:

$$\begin{array}{c} \mathbb{S}_2 F^* \\ \downarrow {\scriptstyle \Delta} \\ F^* \otimes F^* \\ \downarrow {\scriptstyle \delta^{-1}} \\ F \otimes F^* \\ \downarrow {\scriptstyle *} \\ \bigwedge^5 F^* \otimes F^* \\ \downarrow {\scriptstyle \Delta} \\ \bigwedge^3 F^* \otimes \bigwedge^2 F^* \otimes F^* \\ \downarrow {\scriptstyle m_{2,3}} \\ \bigwedge^3 F^* \otimes \bigwedge^3 F^* \\ \downarrow {\scriptstyle \psi^* \otimes \psi^*} \\ V^* \otimes A_1 \end{array}$$

The Betti table for the resolution is

	0	1	2	3	4
total:	1	14	21	14	6
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	14	21	.	.
3:	.	.	.	14	6

7.1.3 The orbit $\mathcal{O}_1$

The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_2]$ is

$$A \leftarrow V_{2\omega_1} \otimes A(-2) \leftarrow V_{\omega_1+\omega_2} \otimes A(-3) \leftarrow V_{\omega_1+\omega_3} \otimes A(-4) \leftarrow$$

$$\leftarrow V_{\omega_1+\omega_3} \otimes A(-6) \leftarrow V_{\omega_1+\omega_2} \otimes A(-7) \leftarrow V_{2\omega_1} \otimes A(-8) \leftarrow A(-10) \leftarrow 0$$

The first differential is precisely the map ρ described in the introduction to this case. The Betti table for the resolution is

	0	1	2	3	4	5	6	7
total:	1	21	64	70	70	64	21	1
0:	1	.	.	.	.	.	.	.
1:	.	21	64	70	.	.	.	.
2:	.	.	.	.	70	64	21	.
3:	.	.	.	.	.	.	.	1

It follows that the orbit closure $\overline{\mathcal{O}}_1$ is Gorenstein.

7.2 The case (F_4, α_2)

The representation is $E \otimes \mathbb{S}_2 F$, where $E = \mathbb{C}^2$ and $F = \mathbb{C}^3$; the group acting is $\mathrm{SL}(E) \times \mathrm{SL}(F) \times \mathbb{C}^\times$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{i;jk} | i = 1, 2; 1 \leqslant j \leqslant k \leqslant 3] = \mathrm{Sym}\,(E^* \otimes \mathbb{S}_2 F^*)\,.$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	4	$x_{1;11} = 1$
$\mathcal{O}_2$	6	$x_{1;12} = 1$
$\mathcal{O}_3$	7	$x_{1;11} = x_{1;23} = 1$
$\mathcal{O}_4$	8	$x_{1;11} = x_{2;22} = 1$
$\mathcal{O}_5$	8	$x_{1;23} = x_{2;13} = 1$
$\mathcal{O}_6$	9	$x_{1;11} = x_{2;22} = x_{123} = 1$
$\mathcal{O}_7$	10	$x_{1;11} = x_{2;23} = 1$
$\mathcal{O}_8$	10	$x_{2;13} = x_{1;23} = x_{1;11} = 1$
$\mathcal{O}_9$	11	$\langle x_{1;11}, x_{1;12}, x_{2;13}, x_{1;22}, x_{2;23} \rangle$
$\mathcal{O}_{10}$	12	$x_{1;11} = x_{2;11} = x_{1;22} = x_{1;33} = 1, x_{2;22} = -1$

To obtain a representative for the orbit $\mathcal{O}_9$ it is enough to assign random rational values to the variables listed, and to set all other variables to 0 (this amounts to taking a generic element in the span of the tensors corresponding to the listed variables).

Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$	$\overline{\mathcal{O}}_3$	$\overline{\mathcal{O}}_4$	$\overline{\mathcal{O}}_5$	$\overline{\mathcal{O}}_6$	$\overline{\mathcal{O}}_7$	$\overline{\mathcal{O}}_8$	$\overline{\mathcal{O}}_9$	$\overline{\mathcal{O}}_{10}$
$\mathcal{O}_0$	ns	s	s	s	s	s	s	s	s	s	ns
$\mathcal{O}_1$		ns	s	ns	s	s	s	s	s	s	ns
$\mathcal{O}_2$			ns	ns	ns	s	s	s	s	s	ns
$\mathcal{O}_3$				ns			s	s	s	s	ns
$\mathcal{O}_4$					ns		s	s	s	s	ns
$\mathcal{O}_5$						ns			s	s	ns
$\mathcal{O}_6$							ns	ns	s	s	ns
$\mathcal{O}_7$								ns		s	ns
$\mathcal{O}_8$									ns	s	ns
$\mathcal{O}_9$										ns	ns
$\mathcal{O}_{10}$											ns

We denote the free A-module $\mathbb{S}_{(a,b)} E^* \otimes \mathbb{S}_{(c,d,e)} F^* \otimes A(-a-b)$ by $(a, b; c, d, e)$.

7.2.1 The orbit $\mathcal{O}_9$

The orbit closure $\overline{\mathcal{O}}_9$ is not normal. The expected resolution for the coordinate ring of the normalization of $\overline{\mathcal{O}}_9$ is

$$A \oplus (1, 1; 2, 1, 1) \oplus (2, 1; 2, 2, 2) \leftarrow (2, 2; 4, 2, 2) \leftarrow 0$$

This orbit closure is a hypersurface, so there is only one differential d_1. The defining equation of $\overline{\mathcal{O}}_9$ is the determinant of d_1. Alternatively this equation can be obtained as the discriminant of the determinant of a generic 3×3 symmetric matrix of linear forms in two variables, which is a homogeneous polynomial of degree 12. Explicitly:

$$\delta = \det \begin{pmatrix} ux_{111} + vx_{211} & ux_{112} + vx_{212} & ux_{113} + vx_{213} \\ ux_{112} + vx_{212} & ux_{122} + vx_{222} & ux_{123} + vx_{223} \\ ux_{113} + vx_{213} & ux_{123} + vx_{223} & ux_{133} + vx_{233} \end{pmatrix} =$$
$$= a_{3,0}u^3 + a_{2,1}u^2v + a_{1,2}uv^2 + a_{0,3}v^3,$$

and

$$\operatorname{disc}(\delta) = 27a_{3,0}^2a_{1,2}^2 + 4a_{3,0}a_{1,2}^3 + 4a_{2,1}^3a_{0,3} - a_{2,1}^2a_{1,2}^2 - 18a_{3,0}a_{2,1}a_{1,2}a_{0,3}.$$

The block $(2,2;4,2,2) \to (2,1;2,2,2)$ of d_1 was constructed as the map

$$\mathbb{S}_2 F^* \xrightarrow{tr^{(1)}} E \otimes E^* \otimes \mathbb{S}_2 F^* \hookrightarrow E \otimes A_1$$

The block $(2,2;4,2,2) \to (1,1;2,1,1)$ was constructed by taking the map

$$\begin{array}{c} \bigwedge^3 F^* \otimes \mathbb{S}_2 F^* \\ \big\downarrow \Delta \otimes \Delta \\ F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \\ \big\downarrow m_{2,4} \otimes m_{3,5} \\ F^* \otimes \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \\ \big\downarrow m_{2,3} \\ F^* \otimes \bigwedge^2 (\mathbb{S}_2 F^*) \end{array}$$

on the F^* factor and then applying the embedding $\bigwedge^2 E^* \otimes \bigwedge^2 (\mathbb{S}_2 F^*) \hookrightarrow A_2$ given by

$$e_1^* \wedge e_2^* \otimes f_i^* f_j^* \wedge f_k^* f_l^* \longmapsto \det \begin{pmatrix} x_{1ij} & x_{1kl} \\ x_{2ij} & x_{2kl} \end{pmatrix}.$$

The last block $(2,2;4,2,2) \to A$ can be obtained as follows. First construct the map

$$
\begin{array}{c}
\bigwedge^3 F^* \otimes \bigwedge^3 F^* \otimes \mathbb{S}_2 F^* \\
\downarrow \scriptstyle{\Delta \otimes \Delta \otimes \Delta} \\
F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \\
\downarrow \scriptstyle{m_{1,7} \otimes m_{2,4} \otimes m_{3,5} \otimes m_{6,8}} \\
\mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \\
\downarrow \scriptstyle{m_{1,2} \otimes m_{3,4}} \\
\bigwedge^2 (\mathbb{S}_2 F^*) \otimes \bigwedge^2 (\mathbb{S}_2 F^*)
\end{array}
$$

on the F^* factor. Then embed into A via the map

$$\bigwedge^2 E^* \otimes \bigwedge^2 (\mathbb{S}_2 F^*) \otimes \bigwedge^2 E^* \otimes \bigwedge^2 (\mathbb{S}_2 F^*) \longrightarrow A_2 \otimes A_2 \longrightarrow A_4$$

where the first step uses the embedding described earlier twice and the second step is symmetric multiplication.

The Betti table for the resolution of the normalization is

	0	1
total:	6	6
0:	1	.
1:	.	.
2:	3	.
3:	2	6

and the Betti table for the resolution of the cokernel $C(9)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_9] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_9)]$ is

	0	1	2
total:	5	6	1
2:	3	.	.
3:	2	6	.
4:	.	.	.
5:	.	.	.
6:	.	.	.
7:	.	.	.
8:	.	.	.
9:	.	.	.
10:	.	.	1

7.2.2 The orbit $\mathcal{O}_8$

The orbit closure $\overline{\mathcal{O}}_8$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_8)$ is

$$A \oplus (1,1;2,1,1) \leftarrow (3,2;4,3,3) \oplus (2,2;4,2,2) \leftarrow (3,3;5,4,3) \leftarrow 0$$

We construct explicitly the differential d_2 as follows. The first block $(3,3;5,4,3) \to (3,2;4,3,3)$ is defined on the F^* factor as

$$F^* \otimes F \xrightarrow{tr^{(1)}} F^* \otimes F \otimes F^* \otimes F \xhookrightarrow{m_{1,3} \otimes m_{2,4}} \mathbb{S}_2 F^* \otimes \bigwedge^2 F$$

restricted to the subspace of traceless tensors $\mathbb{S}_{(2,1)} F^*$. On the E^* factor, it is simply the trace map $\mathbb{C} \to E^* \otimes E$, and putting the factors together we have

$$\mathbb{S}_{(2,1)} F^* \longrightarrow E^* \otimes E \otimes \mathbb{S}_2 F^* \otimes \bigwedge^2 F \hookrightarrow E \otimes \bigwedge^2 F \otimes A_1$$

The second block $(3,3;5,4,3) \to (2,2;4,2,2)$ is defined on the F^* factor as

$$\begin{array}{c} F^* \otimes F \\ \downarrow {\scriptstyle tr^{(1)} \otimes tr^{(1)} \otimes tr^{(1)}} \\ F^* \otimes F \otimes F^* \otimes F \otimes F^* \otimes F \otimes F^* \otimes F \\ \downarrow {\scriptstyle m_{1,3} \otimes m_{2,8} \otimes m_{4,6} \otimes m_{5,7}} \\ \mathbb{S}_2 F^* \otimes \bigwedge^2 F \otimes \bigwedge^2 F \otimes \mathbb{S}_2 F^* \\ \downarrow {\scriptstyle m_{1,4} \otimes * \otimes *} \\ \bigwedge^2 (\mathbb{S}_2 F^*) \otimes F^* \otimes F^* \\ \downarrow {\scriptstyle m_{2,3}} \\ \bigwedge^2 (\mathbb{S}_2 F^*) \otimes \mathbb{S}_2 F^* \end{array}$$

restricted to the subspace of traceless tensors $\mathbb{S}_{(2,1)} F^*$. On the E^* factor, we only have $\bigwedge^2 E^*$, so taking the factors together we get the map

$$\mathbb{S}_{(2,1)} F^* \longrightarrow \bigwedge^2 E^* \otimes \bigwedge^2 (\mathbb{S}_2 F^*) \otimes \mathbb{S}_2 F^* \hookrightarrow \mathbb{S}_2 F^* \otimes A_2$$

where the embedding into A_2 is the one described in 7.2.1. The Betti table for the normalization is

	0	1	2
total:	4	12	8
0:	1	.	.
1:	.	.	.
2:	3	.	.
3:	.	6	.
4:	.	6	8

Dropping the row with entries of degree 4 and 5 in the differential d_1, we obtain a map

$$(3,2;4,3,3) \oplus (2,2;4,2,2) \to (1,1;2,1,1)$$

This is a presentation for the cokernel $C(8)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_8] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_8)]$; the Betti table for $C(8)$ is

	0	1	2	3
total:	3	12	11	2
0:	3	.	.	.
1:	.	6	.	.
2:	.	6	11	.
3:	.	.	.	.
4:	.	.	.	2

Remark 7.2.1. In this particular example, the (truncated) cone procedure in M2 did not produce a result in a reasonable time. However the representation theoretic structure of the resolution of $C(8)$ is easy to determine using the methods in appendix B, or by dimension count. The equivariant form of the resolution is:

$$(1,1;2,1,1) \leftarrow (3,2;4,3,3) \oplus (2,2;4,2,2) \leftarrow$$
$$\leftarrow (3,3;5,4,3) \oplus (4,2;4,4,4) \leftarrow (5,4;6,6,6) \leftarrow 0$$

From this and the resolution of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_8)]$, it is possible to construct and minimize the mapping cone by hand. The resulting complex is

$$(0,0;0,0,0) \leftarrow (4,2;4,4,4) \leftarrow (5,4;6,6,6) \leftarrow 0$$

The last map in the complex can be built as follows:

$$\begin{array}{c} E^* \otimes \bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes \bigwedge^3 F^* \otimes \bigwedge^3 F^* \\ \Big\downarrow \scriptstyle \Delta\otimes\Delta\otimes\Delta\otimes\Delta \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \\ \Big\downarrow \scriptstyle m_{2,4}\otimes m_{6,9}\otimes m_{7,10}\otimes m_{8,11} \\ \mathbb{S}_2 E^* \otimes E^* \otimes E^* \otimes E^* \otimes \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \\ \Big\downarrow \scriptstyle m_{2,5}\otimes m_{3,6}\otimes m_{4,7} \\ \mathbb{S}_2 E^* \otimes A_1 \otimes A_1 \otimes A_1 \\ \Big\downarrow \scriptstyle m_{2,3} \\ \mathbb{S}_2 E^* \otimes A_3 \end{array}$$

and then resolved as usual to obtain the defining equations of $\overline{\mathcal{O}}_8$.

The Betti for the resolution of $\mathbb{C}[\overline{\mathcal{O}}_8]$ is

	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	.	.
2:	.	.	.
3:	.	.	.
4:	.	.	.
5:	.	3	.
6:	.	.	.
7:	.	.	2

It follows that $\overline{\mathcal{O}}_8$ is Cohen-Macaulay.

7.2.3 The orbit $\mathcal{O}_7$

The orbit closure $\overline{\mathcal{O}}_7$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_7)$ is

$$A \oplus (1,1;2,2,0) \leftarrow (2,1;3,2,1) \leftarrow (3,1;3,3,2) \leftarrow 0$$

We construct the differential $d_2 : (3,1;3,3,2) \to (2,1;3,2,1)$ explicitly as follows. On the E^* factor, take the diagonalization $\mathbb{S}_2 E^* \to E^* \otimes E^*$. On the F^* factor, take the map

$$\begin{array}{c} \bigwedge^2 F^* \\ \Big\downarrow {\scriptstyle tr^{(1)}} \\ F \otimes F^* \otimes \bigwedge^2 F^* \\ \Big\downarrow {\scriptstyle \Delta} \\ F \otimes F^* \otimes F^* \otimes F^* \\ \Big\downarrow {\scriptstyle m_{2,3}} \\ F \otimes F^* \otimes \mathbb{S}_2 F^* \end{array}$$

then project the first two factors onto the space of traceless tensors $\mathbb{S}_{(2,1)} F^*$ via the map $F \otimes F^* \to F \otimes F^* / \mathrm{im}(tr^{(1)})$. Altogether we have

$$\mathbb{S}_2 E^* \otimes \bigwedge^2 F^* \longrightarrow E^* \otimes E^* \otimes \mathbb{S}_{(2,1)} F^* \otimes \mathbb{S}_2 F^* \hookrightarrow E^* \otimes \mathbb{S}_{(2,1)} F^* \otimes A_1$$

The Betti table for the normalization is

	0	1	2
total:	7	16	9
0:	1	.	.
1:	.	.	.
2:	6	16	9

Dropping the row of degree 3 in the differential d_1, we obtain a map

$$(2,1;3,2,1) \to (1,1;2,2,0)$$

This is a presentation for the cokernel $C(7)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_7] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_7)]$; the Betti table for $C(7)$ is

	0	1	2	3	4	5
total:	6	16	29	36	21	4
0:	6	16	9	.	.	.
1:	.	.	.	.	.	.
2:	.	.	20	36	21	4

By the cone procedure, we recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_7]$. Its equivariant form is

$$A \leftarrow (3,3;5,5,2) \oplus (3,3;6,3,3) \leftarrow (4,3;5,5,4) \oplus (4,3;6,5,3) \leftarrow$$

$$\leftarrow (4,4;6,5,5) \oplus (5,3;6,6,4) \leftarrow (6,3;6,6,6) \leftarrow 0$$

and it has the following Betti table

	0	1	2	3	4
total:	1	20	36	21	4
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	.	.	.	.
3:	.	.	.	.	.
4:	.	.	.	.	.
5:	.	20	36	21	4

We observe that $\overline{\mathcal{O}}_7$ is not Cohen-Macaulay because it has codimension 2 but its coordinate ring has projective dimension 4.

7.2.4 The orbit $\mathcal{O}_6$

The orbit closure $\overline{\mathcal{O}}_6$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_6)$ is

$$A \oplus (1,1;2,2,0) \leftarrow (2,1;3,2,1) \oplus (2,1;2,2,2) \leftarrow$$

$$\leftarrow (3,3;5,4,3) \oplus (3,1;3,3,2) \leftarrow (4,3;5,5,4) \leftarrow 0$$

We construct the differential

$$d_1 : (2,1;3,2,1) \oplus (2,1;2,2,2) \to (1,1;2,2,0) \oplus A$$

explicitly. Notice how the domain of d_1 is isomorphic to $E^* \otimes F \otimes F^*$. It is clear that the representation on the E^* factor is, up to a power of the determinant, simply E^*. On the F^* factor, we have $\mathbb{S}_{(3,2,1)} F^* \oplus \mathbb{S}_{(2,2,2)} F^*$, with the first summand corresponding to the space of traceless tensors in $F \otimes F^*$ and the second factor corresponding to the tensor with non zero trace.

For the first block $E^* \otimes F \otimes F^* \to (1,1;2,2,0)$, take the map

$$\begin{array}{c} E^* \otimes F \otimes F^* \\ \Big\downarrow{\scriptstyle tr^{(1)}} \\ E^* \otimes F \otimes F^* \otimes F \otimes F^* \\ \Big\downarrow{\scriptstyle m_{2,4} \otimes m_{3,5}} \\ E^* \otimes \mathbb{S}_2 F \otimes \mathbb{S}_2 F^* \\ \Big\downarrow \\ \mathbb{S}_2 F \otimes A_1 \end{array}$$

For the second block $E^* \otimes F \otimes F^* \to A_3$, take the map

$$\begin{array}{c} F \otimes F^* \otimes \bigwedge^3 F^* \\ \Big\downarrow{\scriptstyle *} \\ \bigwedge^2 F^* \otimes F^* \otimes \bigwedge^3 F^* \\ \Big\downarrow{\scriptstyle \Delta \otimes \Delta} \\ F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \\ \Big\downarrow{\scriptstyle m_{1,4} \otimes m_{2,5} \otimes m_{3,6}} \\ \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \otimes \mathbb{S}_2 F^* \\ \Big\downarrow{\scriptstyle m_{1,2}} \\ \bigwedge^2 (\mathbb{S}_2 F^*) \otimes \mathbb{S}_2 F^* \end{array}$$

on the F^* factor. On the E^* factor, simply tensor by a power of the determinant. Altogether we have

$$E^* \otimes \bigwedge^2 E^* \otimes \bigwedge^2 (\mathbb{S}_2 F^*) \otimes \mathbb{S}_2 F^* \hookrightarrow A_1 \otimes A_2 \longrightarrow A_3$$

where the first step uses the embedding defined in 7.2.1 and the second step is symmetric multiplication.

The Betti table for the normalization is

	0	1	2	3
total:	7	18	17	6
0:	1	.	.	.
1:	.	.	.	.
2:	6	18	9	.
3:	.	.	.	.
4:	.	.	8	6

Dropping the row of degree 3 in the differential d_1, we obtain a map

$$(2,1;3,2,1) \oplus (2,1;2,2,2) \to (1,1;2,2,0)$$

This is a presentation for the cokernel $C(6)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_6] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_6)]$; the Betti table for $C(6)$ is

	0	1	2	3	4
total:	6	18	15	6	3
2:	6	18	15	.	.
3:	.	.	.	.	.
4:	.	.	.	6	3

By the cone procedure, we recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_6]$. Its equivariant form is

$$A \leftarrow (2,2;4,2,2) \leftarrow (3,3;5,4,3) \leftarrow (4,4;6,5,5) \leftarrow 0$$

and it has the following Betti table

	0	1	2	3
total:	1	6	8	3
0:	1	.	.	.
1:	.	.	.	.
2:	.	.	.	.
3:	.	6	.	.
4:	.	.	8	.
5:	.	.	.	3

We observe that $\overline{\mathcal{O}}_6$ is Cohen-Macaulay.

7.2.5 The orbit $\mathcal{O}_5$

The orbit closure $\overline{\mathcal{O}}_5$ is not normal. The expected resolution for the coordinate ring of the normalization $\mathcal{N}(\overline{\mathcal{O}}_5)$ is

$$A \oplus (1,0;1,1,0) \leftarrow (3,0;2,2,2) \oplus (1,1;3,1,0) \oplus (2,0;2,1,1) \leftarrow$$
$$\leftarrow (3,1;4,2,2) \oplus (2,2;3,3,2) \oplus (2,1;4,1,1) \leftarrow$$
$$\leftarrow (3,2;5,3,2) \leftarrow (3,3;5,5,2) \leftarrow 0$$

We construct the differential $d_4 : (3,3;5,5,2) \to (3,2;5,3,2)$ explicitly. Start with the map

$$\begin{array}{c} \mathbb{S}_3 F \\ \Big\downarrow {\scriptstyle tr^{(1)} \otimes tr^{(1)} \otimes tr^{(1)}} \\ E \otimes E^* \otimes \mathbb{S}_3 F \otimes F \otimes F^* \otimes F \otimes F^* \\ \Big\downarrow {\scriptstyle m_{4,6} \otimes m_{5,7}} \\ E \otimes E^* \otimes \mathbb{S}_3 F \otimes \mathbb{S}_2 F \otimes \mathbb{S}_2 F^* \\ \Big\downarrow \\ E \otimes \mathbb{S}_3 F \otimes \mathbb{S}_2 F \otimes A_1 \end{array}$$

Then project onto $E \otimes \mathbb{S}_{(3,2)} F \otimes A_1$ modding out $\mathbb{S}_3 F \otimes \mathbb{S}_2 F$ by the image of the map

$$\begin{array}{c} \mathbb{S}_4 F \otimes F \\ \big\downarrow \Delta \\ \mathbb{S}_3 F \otimes F \otimes F \\ \big\downarrow m_{2,3} \\ \mathbb{S}_2 F \end{array}$$

The Betti table for the normalization is

	0	1	2	3	4
total:	7	28	41	30	10
0:	1	.	.	.	.
1:	6	24	20	.	.
2:	.	4	21	30	10

Dropping the row of degree 2 in the differential d_1, we obtain a map

$$(3,0;2,2,2) \oplus (1,1;3,1,0) \oplus (2,0;2,1,1) \to (1,0;1,1,0)$$

This is a presentation for the cokernel $C(5)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_5] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_5)]$; the Betti table for $C(5)$ is

	0	1	2	3	4	5	6	7	8
total:	6	24	56	163	298	285	144	33	1
1:	6	24	20	.	.	.	.	.	.
2:	.	.	36	42	10	.	.	.	.
3:	.	.	.	121	288	285	144	33	.
4:	.	.	.	.	.	.	.	.	1

By the cone procedure, we recover the resolution of $\mathbb{C}[\overline{\mathcal{O}}_5]$. Its equivariant form is

$$A \leftarrow (3,0;2,2,2) \oplus (2,2;4,4,0) \leftarrow$$

$$\leftarrow (3,3;6,5,1) \oplus (3,2;4,4,2) \oplus (4,2;6,4,2) \oplus (5,1;4,4,4) \leftarrow$$

$$\leftarrow (4,3;6,6,2) \oplus (4,3;6,5,3) \oplus (4,3;6,4,4) \oplus (4,3;7,5,2) \oplus$$

$$\oplus (4,3;7,4,3) \oplus (5,2;6,5,3) \oplus (5,2;6,4,4) \leftarrow$$

$$\leftarrow (4,4;6,6,4) \oplus (4,4;6,5,5) \oplus (4,4;7,6,3) \oplus (4,4;7,5,4) \oplus (4,4;8,5,3) \oplus$$

$$\oplus (5,3;6,6,4) \oplus (5,3;7,6,3) \oplus 2 * (5,3;7,5,4) \oplus (6,2;6,5,5) \leftarrow$$

$$\leftarrow (5,4;6,6,6) \oplus (5,4;7,7,4) \oplus (5,4;7,6,5) \oplus$$

$$\oplus (5,4;8,6,4) \oplus (5,4;8,5,5) \oplus (6,3;7,6,5) \leftarrow$$

$$\leftarrow (5,5;8,7,5) \oplus (6,4;8,6,6) \leftarrow (6,6;8,8,8) \leftarrow 0$$

and it has the following Betti table

	0	1	2	3	4	5	6	7
total:	1	19	133	288	285	144	33	1
0:	1	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.
2:	.	4	.	.	.	.	.	.
3:	.	15	12	.	.	.	.	.
4:	.	.	121	288	285	144	33	.
5:	.	.	.	.	.	.	.	1

We observe that $\overline{\mathcal{O}}_5$ is not Cohen-Macaulay because it has codimension 4 but its coordinate ring has projective dimension 7.

7.2.6 The orbit $\mathcal{O}_4$

The orbit closure $\overline{\mathcal{O}}_4$ is normal with rational singularities. The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_4]$ is

$$A \leftarrow (3,0;2,2,2) \oplus (2,1;3,2,1) \leftarrow$$
$$\leftarrow (2,2;3,3,2) \oplus (3,1;3,3,2) \oplus (3,1;4,2,2) \oplus (2,2;4,3,1) \leftarrow$$
$$\leftarrow (3,2;5,3,2) \oplus (3,2;4,3,3) \leftarrow (3,3;6,3,3) \leftarrow 0$$

The differential d_4 was written explicitly. For the block, $(3,3;6,3,3) \to (3,2;4,3,3)$ take the map

$$\mathbb{S}_3\, F^* \xrightarrow{tr^{(1)} \otimes \Delta} E \otimes E^* \otimes \mathbb{S}_2\, F^* \otimes F^* \hookrightarrow E \otimes F^* \otimes A_1$$

For the block $(3,3;6,3,3) \to (3,2;5,3,2)$ construct first the map

$$\begin{array}{c} \mathbb{S}_3\, F^* \otimes \bigwedge^3 F^* \\ \downarrow {\scriptstyle tr^{(1)} \otimes \Delta \otimes \Delta} \\ E \otimes E^* \otimes \mathbb{S}_2\, F^* \otimes F^* \otimes F^* \otimes F^* \otimes F^* \\ \downarrow {\scriptstyle m_{3,5} \otimes m_{4,6}} \\ E \otimes E^* \otimes \mathbb{S}_3\, F^* \otimes \mathbb{S}_2\, F^* \otimes F^* \\ \downarrow \\ E \otimes \mathbb{S}_3\, F^* \otimes F^* \otimes A_1 \end{array}$$

Finally project onto $E \otimes \mathbb{S}_{(3,1)}\, F^* \otimes A_1$ by modding out $\mathbb{S}_3\, F^* \otimes F^*$ by the image of the map

$$\mathbb{S}_4\, F^* \xrightarrow{\Delta} \mathbb{S}_3\, F^* \otimes F^*$$

The Betti table for the resolution is

	0	1	2	3	4
total:	1	20	45	36	10
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	20	45	36	10

We conclude that $\overline{\mathcal{O}}_4$ is Cohen-Macaulay. The defining equations are the 3×3 minors of the generic matrix of a linear map $E \otimes F \to F^*$ after symmetrizing indices on the F^* side.

7.2.7 The orbit $\mathcal{O}_3$

The orbit closure $\overline{\mathcal{O}}_3$ is normal with rational singularities. It is degenerate with equations given by the 2×2 minors of

$$\begin{pmatrix} x_{111} & x_{211} \\ x_{112} & x_{212} \\ x_{113} & x_{213} \\ x_{122} & x_{222} \\ x_{123} & x_{223} \\ x_{133} & x_{233} \end{pmatrix}$$

the generic matrix of a linear map $E \to \mathbb{S}_2 F^*$. The Betti table for the resolution is

	0	1	2	3	4	5
total:	1	15	40	45	24	5
0:	1	.	.	.	.	.
1:	.	15	40	45	24	5

It follows that $\overline{\mathcal{O}}_3$ is Cohen-Macaulay.

7.2.8 The orbit $\mathcal{O}_2$

The orbit closure $\overline{\mathcal{O}}_2$ is normal with rational singularities. It is degenerate with equations given by the 2×2 minors of the generic matrix of a linear map $E \to \mathbb{S}_2 F^*$ together with the coefficients of the determinant of a generic 3×3 matrix of linear forms in two variables. The former are the equations of $\overline{\mathcal{O}}_3$ while the latter are the coefficients $a_{3,0}, a_{2,1}, a_{1,2}, a_{0,3}$ of δ as defined in 7.2.1. The Betti table for the resolution is

	0	1	2	3	4	5	6
total:	1	19	58	75	44	11	2
0:	1	.	.	.	.	.	.
1:	.	15	40	45	24	5	.
2:	.	4	18	30	20	.	.
3:	.	.	.	.	.	6	2

It follows that $\overline{\mathcal{O}}_2$ is Cohen-Macaulay.

7.2.9 The orbit $\mathcal{O}_1$

The orbit closure $\overline{\mathcal{O}}_1$ is normal with rational singularities. It is degenerate with equations given by the 2×2 minors of the generic matrix of a linear map $E \to \mathbb{S}_2 F^*$ together with the coefficients of the 2×2 minors of a generic 3×3 matrix of linear forms in two variables. The former are the equations of $\overline{\mathcal{O}}_3$ while the latter are the coefficients of the 2×2 minors of the matrix defined in 7.2.1. The Betti table for the resolution is

	0	1	2	3	4	5	6	7	8
total:	1	33	144	294	336	210	69	16	3
0:	1	.	.	.	.	.	.	.	.
1:	.	33	144	294	336	210	48	.	.
2:	.	.	.	.	.	.	21	16	3

It follows that $\overline{\mathcal{O}}_1$ is Cohen-Macaulay.

7.3 The case (F_4, α_3)

The representation is $E\otimes F$, where $E = \mathbb{C}^2$ and $F = \mathbb{C}^3$; the group acting is $\mathrm{SL}(E)\times\mathrm{SL}(F)\times\mathbb{C}^\times$. The orbit closures for this representation are classical determinantal varieties. For a description of the minimal free resolutions of their coordinate rings, the reader can consult [20, Ch. 6], for example.

7.4 The case (F_4, α_4)

The representation is $V = V(\omega_3, B_3)$, the third fundamental representation of the group $\mathrm{SO}(7, \mathbb{C})$; the group acting is Spin(7). The corresponding polynomial ring is

$$A = \mathbb{C}[x_{1231}, x_{1221}, x_{1121}, x_{0121}, x_{1111}, x_{0111}, x_{0011}, x_{0001}] = \mathrm{Sym}(V^*).$$

The variables in A are indexed by the roots in $\mathfrak{g}_1$. The variables are weight vectors in V^*, with the following weights:

$$x_{1231} \leftrightarrow \tfrac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3) \qquad x_{1221} \leftrightarrow \tfrac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3)$$

$$x_{1121} \leftrightarrow \tfrac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3) \qquad x_{0121} \leftrightarrow \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3)$$

$$x_{1111} \leftrightarrow \tfrac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3) \qquad x_{0111} \leftrightarrow \tfrac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3)$$

$$x_{0011} \leftrightarrow \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 + \epsilon_3) \qquad x_{0001} \leftrightarrow \tfrac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3)$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	7	$x_{0001} = 1$
$\mathcal{O}_2$	8	$x_{0001} = x_{1231} = 1$

All the orbit closures are normal, Cohen-Macaulay, Gorenstein and have rational singularities. Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$
$\mathcal{O}_0$	ns	s	ns
$\mathcal{O}_1$		ns	ns
$\mathcal{O}_2$			ns

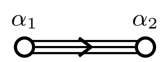
7.4.1 The orbit $\mathcal{O}_1$

The variety $\overline{\mathcal{O}}_1$ is the closure of the highest weight vector orbit. It is a hypersurface defined by an invariant of degree 2. The invariant was described explicitly by Igusa [8].

Chapter 8

Representations of type G_2

In this chapter, we will analyze the cases corresponding to gradings on the simple Lie algebra of type G_2. Each case corresponds to the choice of a distinguished node on the Dynkin diagram for G_2. The nodes are numbered according to the conventions in Bourbaki [1].



8.1 The case (G_2, α_1)

The representation is $E = \mathbb{C}^2$ and the group acting is $\mathrm{GL}(E)$. The only orbits in this case are the origin and the dense orbit.

8.2 The case (G_2, α_2)

The representation is $\mathbb{S}_3 E$, where $E = \mathbb{C}^2$; the group acting is $\mathrm{GL}(E)$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{ijk} \mid 1 \leqslant i \leqslant j \leqslant 2] = \mathrm{Sym}(\mathbb{S}_3 E^*).$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	2	$x_{111} = 1$
$\mathcal{O}_2$	3	$x_{112} = 1$
$\mathcal{O}_3$	4	$x_{111} = x_{222} = 1$

Here is the containment and singularity table:

	$\overline{\mathcal{O}}_0$	$\overline{\mathcal{O}}_1$	$\overline{\mathcal{O}}_2$	$\overline{\mathcal{O}}_3$
$\mathcal{O}_0$	ns	s	s	ns
$\mathcal{O}_1$		ns	s	ns
$\mathcal{O}_2$			ns	ns
$\mathcal{O}_3$				ns

We denote the free A-module $\mathbb{S}_{(a,b)} E^* \otimes A(-(a+b)/3)$ by (a,b).

8.2.1 The orbit $\mathcal{O}_2$

The orbit closure $\overline{\mathcal{O}}_2$ is not normal. The expected resolution for the coordinate ring of the normalization of $\overline{\mathcal{O}}_2$ is

$$A \oplus (2,1) \leftarrow (4,2) \leftarrow 0$$

The orbit closure is a hypersurface, so there is only one differential d_1. The defining equation of $\overline{\mathcal{O}}_9$ is the determinant of d_1. Alternatively this equation can be obtained using the cone procedure.

The block $(4,2) \to (2,1)$ of d_1 was constructed by taking the map

$$\bigwedge^2 E^* \otimes \mathbb{S}_2 E^* \xrightarrow{\Delta \otimes \Delta} E^* \otimes E^* \otimes E^* \otimes E^* \xrightarrow{m_{2,3,4}} E^* \otimes A_1$$

The block $(4,2) \to A$ can be obtained as follows:

$$\begin{array}{c}
\bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes \mathbb{S}_2 E^* \\
\Big\downarrow {\scriptstyle \Delta \otimes \Delta \otimes \Delta} \\
E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \\
\Big\downarrow {\scriptstyle m_{1,3,5} \otimes m_{2,4,6}} \\
A_1 \otimes A_1 \\
\Big\downarrow \\
A_2
\end{array}$$

where the last step is symmetric multiplication.

The Betti table for the resolution of the normalization is

	0	1
total:	3	3
0:	1	.
1:	2	3

and the Betti table for the resolution of the cokernel $C(2)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_2] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_2)]$ is

	0	1	2
total:	2	3	1
1:	2	3	.
2:	.	.	1

8.2.2 The orbit $\mathcal{O}_1$

The orbit closure $\overline{\mathcal{O}}_1$ is normal with rational singularities. The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_1]$ is

$$A \leftarrow (4,2) \leftarrow (5,4) \leftarrow 0$$

Notice that the first differential is the second block of the differential described in 8.2.1. Here we describe how to explicitly construct $d_2 : (5,4) \to (4,2)$, the second differential in the resolution. This is obtained by writing the map

$$\begin{array}{c} \bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes E^* \\ \Big\downarrow {\scriptstyle \Delta \otimes \Delta} \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \\ \Big\downarrow {\scriptstyle m_{2,3} \otimes m_{1,4,5}} \\ \mathbb{S}_2\, E^* \otimes A_1 \end{array}$$

The Betti table for the resolution is

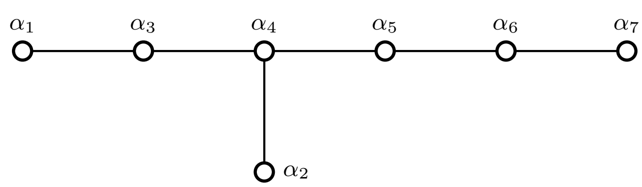
	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	3	2

It follows that the orbit closure $\overline{\mathcal{O}}_1$ is Cohen-Macaulay.

Chapter 9

Representations of type E_7

In this chapter, we will analyze some cases corresponding to gradings on the simple Lie algebra of type E_7. Each case corresponds to the choice of a distinguished node on the Dynkin diagram for E_7. The nodes are numbered according to the conventions in Bourbaki [1].



9.1 The case (E_7, α_2)

The representation is $\bigwedge^3 F$, where $F = \mathbb{C}^7$; the group acting is $\mathrm{GL}(F)$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{ijk} | 1 \leqslant i < j < k \leqslant 7] = \mathrm{Sym}\left(\bigwedge^3 F^*\right).$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	13	$x_{123} = 1$
$\mathcal{O}_2$	20	$x_{123} = x_{145} = 1$
$\mathcal{O}_3$	21	$x_{123} = x_{145} = x_{167} = 1$
$\mathcal{O}_4$	25	$x_{123} = x_{145} = x_{246} = 1$
$\mathcal{O}_5$	26	$x_{123} = x_{456} = 1$
$\mathcal{O}_6$	28	$x_{123} = x_{145} = x_{167} = x_{357} = 1$
$\mathcal{O}_7$	31	$x_{123} = x_{456} = x_{147} = 1$
$\mathcal{O}_8$	34	$x_{123} = x_{456} = x_{147} = x_{257} = 1$
$\mathcal{O}_9$	35	$x_{123} = x_{456} = x_{147} = x_{257} = x_{367} = 1$

We denote the free A-module $\mathbb{S}_\lambda F^* \otimes A(-|\lambda|/3)$ by (λ).

9.1.1 The orbit $\mathcal{O}_7$

The orbit closure $\overline{\mathcal{O}}_7$ is normal with rational singularities. The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_7]$ is

$$(0^7) \leftarrow (3^4, 2^3) \leftarrow (4, 3^5, 2) \leftarrow (5^2, 4^5) \leftarrow (6, 5^6) \leftarrow 0$$

The differential $d_2 : (4, 3^5, 2) \to (3^4, 2^3)$ was written explicitly as the map

$$\bigwedge^6 F^* \otimes F^* \xrightarrow{\Delta} \bigwedge^4 F^* \otimes \bigwedge^2 F^* \otimes F^* \xrightarrow{m_{2,3}} \bigwedge^4 F^* \otimes A_1$$

restricted to the kernel of the exterior multiplication $\bigwedge^6 F^* \otimes F^* \to \bigwedge^7 F^*$, i.e. the subspace of traceless 7×7 matrices. The Betti table for the resolution is

	0	1	2	3	4
total:	1	35	48	21	7
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	.	.	.	.
3:	.	.	.	.	.
4:	.	.	.	.	.
5:	.	35	48	.	.
6:	.	.	.	.	.
7:	.	.	.	21	.
8:	.	.	.	.	7

We conclude that $\overline{\mathcal{O}}_7$ is Cohen-Macaulay. Using the equations from the differential d_1, we can also observe that $\overline{\mathcal{O}}_7 = \mathcal{O}_0 \cup \ldots \cup \mathcal{O}_7$ and its singular locus is $\mathcal{O}_0 \cup \ldots \cup \mathcal{O}_6$.

9.1.2 The orbit $\mathcal{O}_6$

The orbit closure $\overline{\mathcal{O}}_6$ is normal with rational singularities. The expected resolution for the coordinate ring $\mathbb{C}[\overline{\mathcal{O}}_6]$ is

$$(0^7) \leftarrow (2^6,0) \leftarrow (3^5,2,1) \leftarrow (4^4,3^2,2) \leftarrow (5^3,4^3,3) \leftarrow$$
$$\leftarrow (6^2,5^4,4) \leftarrow (7,6^5,5) \leftarrow (7^6,6) \leftarrow 0$$

The differential $d_7 : (7^6,6) \to (7,6^5,5)$ was written as follows. First construct the equivariant map

$$\bigwedge^7 F^* \otimes \bigwedge^6 F^* \xrightarrow{\Delta\otimes\Delta} \bigwedge^6 F^* \otimes F^* \otimes \bigwedge^5 F^* \otimes F^*$$

The second and third factor are combined into elements of $\mathbb{S}_2\left(\bigwedge^3 F^*\right) = A_2$ via the map

$$F^* \otimes \bigwedge^5 F^* \xrightarrow{\Delta} F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^3 F^* \xrightarrow{m_{1,2}} \bigwedge^3 F^* \otimes \bigwedge^3 F^* \xrightarrow{m_{1,2}} A_2$$

Altogether this gives a map $\bigwedge^7 F^* \otimes \bigwedge^6 F^* \to \bigwedge^6 F^* \otimes F^* \otimes A_2$. Finally we restrict to the subspace of traceless 7×7 matrices, via the map

$$\bigwedge^6 F^* \otimes F^* \xrightarrow{\mathrm{id}-\frac{1}{7}ip} \ker\left(\bigwedge^6 F^* \otimes F^* \longrightarrow \bigwedge^7 F^*\right)$$

where $p : \bigwedge^6 F^* \otimes F^* \longrightarrow \bigwedge^7 F^*$ is exterior multiplication, and $i : \bigwedge^7 F^* \longrightarrow \bigwedge^6 F^* \otimes F^*$ is the diagonal map (see appendix A). The Betti table for the resolution is

	0	1	2	3	4	5	6	7
total:	1	28	112	210	224	140	48	7
0:	1	.	.	.	.	.	.	.
1:	.	.	.	.	.	.	.	.
2:	.	.	.	.	.	.	.	.
3:	.	28	.	.	.	.	.	.
4:	.	.	112	.	.	.	.	.
5:	.	.	.	210	.	.	.	.
6:	.	.	.	.	224	.	.	.
7:	.	.	.	.	.	140	.	.
8:	.	.	.	.	.	.	48	.
9:	.	.	.	.	.	.	.	7

We conclude that $\overline{\mathcal{O}}_6$ is Cohen-Macaulay. Using the equations from the differential d_1, we can also observe that $\overline{\mathcal{O}}_6 = \mathcal{O}_0 \cup \ldots \cup \mathcal{O}_4 \cup \mathcal{O}_6$ and its singular locus is $\mathcal{O}_0 \cup \ldots \cup \mathcal{O}_4$.

9.2 The case (E_7, α_3)

The representation is $E \otimes \bigwedge^2 F$, where $E = \mathbb{C}^2$ and $F = \mathbb{C}^6$; the group acting is $\mathrm{SL}(E) \times \mathrm{SL}(F) \times \mathbb{C}^\times$. The corresponding polynomial ring is

$$A = \mathbb{C}[x_{a;ij} | a = 1, 2; 1 \leqslant i < j \leqslant 6] = \mathrm{Sym}\left(E^* \otimes \bigwedge^2 F^*\right).$$

In characteristic zero, the representation has the following orbits, listed along with the dimension of the closure and a representative:

orbit	dimension	representative
$\mathcal{O}_0$	0	0
$\mathcal{O}_1$	10	$x_{1;12} = 1$
$\mathcal{O}_2$	15	$x_{1;12} = 1, x_{1;34} = 1$
$\mathcal{O}_3$	15	$x_{1;12} = 1, x_{2;13} = 1$
$\mathcal{O}_4$	16	$x_{1;12} = 1, x_{1;34} = 1, x_{1;56} = 1$
$\mathcal{O}_5$	19	$x_{1;12} = 1, x_{1;34} = 1, x_{2;13} = 1$
$\mathcal{O}_6$	20	$x_{1;12} = 1, x_{2;34} = 1$
$\mathcal{O}_7$	23	$x_{1;12} = 1, x_{2;34} = 1, x_{1;35} = 1$
$\mathcal{O}_8$	25	$x_{1;12} = 1, x_{2;34} = 1, x_{1;35} = 1, x_{2;15} = 1$
$\mathcal{O}_9$	24	$x_{1;12} = 1, x_{2;34} = 1, x_{1;35} = 1, x_{1;46} = 1$
$\mathcal{O}_{10}$	26	$x_{1;12} = 1, x_{2;34} = 1, x_{1;45} = 1, x_{2;16} = 1$
$\mathcal{O}_{11}$	28	$x_{1;12} = 1, x_{2;34} = 1, x_{1;45} = 1, x_{2;16} = 1, x_{1;36} = 1$
$\mathcal{O}_{12}$	25	$x_{1;12} = 1, x_{2;34} = 1, x_{1;56} = 1$
$\mathcal{O}_{13}$	29	$x_{1;12} = 1, x_{2;34} = 1, x_{1;56} = 1, x_{2;15} = 1$
$\mathcal{O}_{14}$	30	$\langle x_{1;12}, x_{2;12}, x_{1;34}, x_{2;34}, x_{1;56}, x_{2;56} \rangle$

To obtain a representative for the orbit $\mathcal{O}_{14}$ it is enough to assign random rational values to the variables listed, and to set all other variables to 0 (this amounts to taking a generic element in the span of the tensors corresponding to the listed variables).

We denote the free A-module $\mathbb{S}_{(a,b)} E^* \otimes \mathbb{S}_{(c,d,e,f,g,h)} F^* \otimes A(-a-b)$ by $(a, b; c, d, e, f, g, h)$.

9.2.1 The orbit $\mathcal{O}_{13}$

The orbit closure $\overline{\mathcal{O}}_{13}$ is not normal. The expected resolution for the coordinate ring of the normalization of $\overline{\mathcal{O}}_{13}$ is

$$A \oplus (2, 1; 1^6) \leftarrow (4, 2; 2^6) \leftarrow 0$$

The differential d_1 can be constructed explicitly as follows. The first block $(4, 2; 2^6) \to (2, 1; 1^6)$ was constructed by taking the map

$$\begin{array}{c} \bigwedge^2 E^* \otimes \mathbb{S}_2 E^* \otimes \bigwedge^6 F^* \\ \big\downarrow {\scriptstyle \Delta \otimes \Delta \otimes \Delta} \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\ \big\downarrow {\scriptstyle m_{2,5} \otimes m_{3,6} \otimes m_{4,7}} \\ E^* \otimes A_1 \otimes A_1 \otimes A_1 \\ \big\downarrow \\ E^* \otimes A_3 \end{array}$$

and the last map is symmetric multiplication. The second block could not be constructed directly; however it appears in the resolution of $\mathbb{C}[\overline{\mathcal{O}}_{11}]$, which we describe in 9.2.3.

The Betti table for the resolution of the normalization is

	0	1
total:	3	3
0:	1	.
1:	.	.
2:	.	.
3:	2	.
4:	.	.
5:	.	3

and the Betti table for the resolution of the cokernel $C(13)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{13}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{13})]$ is

	0	1	2
total:	2	3	1
3:	2	.	.
4:	.	.	.
5:	.	3	.
6:	.	.	.
7:	.	.	.
8:	.	.	.
9:	.	.	.
10:	.	.	1

The defining equation of $\overline{\mathcal{O}}_{13}$ can be obtained using the cone procedure. Alternatively, it could be obtained by taking the determinant of d_1 in the resolution of the normalization (although this method did not produce a result in a reasonable time). Using the defining equation, it is easy to see that $\overline{\mathcal{O}}_{13} = \mathcal{O}_{13} \cup \ldots \cup \mathcal{O}_0$ and its singular locus is $\mathcal{O}_{12} \cup \ldots \cup \mathcal{O}_0$.

9.2.2 The orbit $\mathcal{O}_{12}$

The orbit closure $\overline{\mathcal{O}}_{12}$ is not normal. The expected resolution for the coordinate ring of the normalization of $\overline{\mathcal{O}}_{12}$ is

$$A \oplus (1,1;1^4,0^2) \leftarrow (2,1;2,1^4,0) \leftarrow (4,1;2^5,0) \oplus (3,1;3,1^5) \leftarrow$$
$$\leftarrow (5,1;3,2^4,1) \leftarrow (6,1;3^2,2^4) \leftarrow (8,1;3^6) \leftarrow 0$$

The differential $d_5 : (8,1;3^6) \to (6,1;3^2,2^4)$ was written explicitly as the map

$$\begin{array}{c} \mathbb{S}_7\, E^* \otimes \bigwedge^6 F^* \\ \big\downarrow {\scriptstyle \Delta \otimes \Delta} \\ \mathbb{S}_5\, E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\ \big\downarrow {\scriptstyle m_{2,5} \otimes m_{3,6}} \\ \mathbb{S}_5\, E^* \otimes \bigwedge^2 F^* \otimes A_1 \otimes A_1 \\ \big\downarrow \\ \mathbb{S}_5\, E^* \otimes A_2 \end{array}$$

and the last map is given by symmetric multiplication.

The Betti table for the resolution of the normalization is

	0	1	2	3	4	5
total:	16	70	147	175	90	8
0:	1	.	.	.	.	.
1:	.	.	.	.	.	.
2:	15	70	63	.	.	.
3:	.	.	84	175	90	.
4:	.	.	.	.	.	8

Dropping the row of degree 3 in the differential d_1, we obtain a map $(2,1;2,1^4,0) \to (1,1;1^4,0^2)$, which is a presentation for the cokernel $C(12)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_{12}] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_{12})]$. The resolution of $C(12)$ could only be calculated up to the second syzygies. The equivariant form of this partial resolution is:

$$(1,1;1^4,0^2) \leftarrow (2,1;2,1^4,0) \leftarrow$$
$$\leftarrow (4,1;2^5,0) \oplus (3,1;3,1^5) \oplus (3,3;3^3,1^3) \oplus (3,3;3^2,2^3,0) \oplus (3,3;4,2^3,1^2)$$

and it has Betti table

	0	1	2
total:	15	70	882
2:	15	70	63
3:	.	.	84
4:	.	.	735

By constructing a partial cone by hand, we can observe that the defining ideal of $\overline{\mathcal{O}}_{12}$ is generated by the representation $(3,3;3^3,1^3) \oplus (3,3;3^2,2^3,0) \oplus (3,3;4,2^3,1^2)$, for a total of 735 equations of degree 6.

Following remark 4.2.1, we find that $\overline{\mathcal{O}}_{12} = \mathcal{O}_{12} \cup \mathcal{O}_9 \cup \mathcal{O}_7 \cup \ldots \cup \mathcal{O}_0$.

9.2.3 The orbit $\mathcal{O}_{11}$

The orbit closure $\mathcal{O}_{11}$ is normal with rational singularities. The expected resolution for the coordinate ring of $\overline{\mathcal{O}}_{11}$ is

$$A \leftarrow (4,2;2^6) \leftarrow (5,4;3^6) \leftarrow 0$$

The differential $d_2 : (5,4;3^6) \to (4,2;2^6)$ was written explicitly as the map

$$\begin{array}{c} \bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes E^* \otimes \bigwedge^6 F^* \\ \big\downarrow {\scriptstyle \Delta \otimes \Delta \otimes \Delta} \\ E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\ \big\downarrow {\scriptstyle m_{2,4} \otimes m_{1,6} \otimes m_{3,7} \otimes m_{5,8}} \\ \mathbb{S}_2 E^* \otimes A_1 \otimes A_1 \otimes A_1 \\ \big\downarrow \\ \mathbb{S}_2 E^* \otimes A_3 \end{array}$$

where the last map is symmetric multiplication. The Betti table for the resolution is

	0	1	2
total:	1	3	2
0:	1	.	.
1:	.	.	.
2:	.	.	.
3:	.	.	.
4:	.	.	.
5:	.	3	.
6:	.	.	.
7:	.	.	2

Given the length of the resolution and the ranks of its free modules, the structure of the resolution is completely described by the Hilbert-Burch theorem [2]. In particular, the entries of d_1 are the maximal minors of d_2 with the appropriate sign.

We conclude that $\overline{\mathcal{O}}_{11}$ is Cohen-Macaulay. Using the equations from the differential d_1, we can observe that $\overline{\mathcal{O}}_{11} = \mathcal{O}_{11} \cup \ldots \cup \mathcal{O}_0$ and its singular locus is $\mathcal{O}_{10} \cup \ldots \cup \mathcal{O}_0$.

9.2.4 The orbit $\mathcal{O}_{10}$

The orbit closure $\mathcal{O}_{10}$ is normal with rational singularities. The expected resolution for the coordinate ring of $\overline{\mathcal{O}}_{10}$ is

$$A \leftarrow (3,0;1^6) \leftarrow (3,3;2^6) \oplus (5,1;2^6) \leftarrow (6,3;3^6) \leftarrow (6,6;4^6) \leftarrow 0$$

The differential $d_1 : (3,0;1^6) \to A$ was written explicitly as the map

$$\begin{array}{c} \mathbb{S}_3 E^* \otimes \bigwedge^6 F^* \\ \Big\downarrow {\scriptstyle \Delta \otimes \Delta} \\ E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\ \Big\downarrow {\scriptstyle m_{1,4} \otimes m_{2,5} \otimes m_{3,6}} \\ A_1 \otimes A_1 \otimes A_1 \\ \Big\downarrow \\ A_3 \end{array}$$

and the last map is symmetric multiplication.

The Betti table for the resolution is

	0	1	2	3	4
total:	1	4	6	4	1
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	4	.	.	.
3:	.	.	.	.	.
4:	.	.	6	.	.
5:	.	.	.	.	.
6:	.	.	.	4	.
7:	.	.	.	.	.
8:	.	.	.	.	1

We conclude that $\overline{\mathcal{O}}_{10}$ is Gorenstein. Using the equations from the differential d_1, we can observe that $\overline{\mathcal{O}}_{10} = \mathcal{O}_{10} \cup \mathcal{O}_8 \cup \ldots \cup \mathcal{O}_5 \cup \mathcal{O}_3 \cup \ldots \cup \mathcal{O}_0$ and its singular locus is $\mathcal{O}_7 \cup \ldots \cup \mathcal{O}_5 \cup \mathcal{O}_3 \cup \ldots \cup \mathcal{O}_0$.

9.2.5 The orbit $\mathcal{O}_9$

The orbit closure $\overline{\mathcal{O}}_9$ is not normal. The expected resolution for the coordinate ring of the normalization of $\overline{\mathcal{O}}_9$ is

$$A \oplus (1,1;1^4,0^2) \leftarrow (2,1;1^6) \oplus (2,1;2,1^4,0) \leftarrow$$
$$\leftarrow (3,1;3,1^5) \oplus (4,1;2^5,0) \oplus (3,3;3,2^4,1) \leftarrow$$
$$\leftarrow (5,1;3,2^4,1) \oplus (4,3;4,2^5) \oplus (5,3;3^5,1) \leftarrow (6,1;3^2,2^4) \oplus (6,3;4,3^4,2) \leftarrow$$
$$\leftarrow (8,1;3^6) \oplus (7,3;4^2,3^4) \leftarrow (9,3;4^6) \leftarrow 0$$

The differential d_6 was written explicitly. The block $(9,3;4^6) \rightarrow (7,3;4^2,3^4)$ was constructed as follows:

$$\begin{array}{c}
\mathbb{S}_6 E^* \otimes \bigwedge^6 F^* \\
\big\downarrow {\scriptstyle \Delta\otimes\Delta} \\
\mathbb{S}_4 E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\
\big\downarrow {\scriptstyle m_{2,5}\otimes m_{3,6}} \\
\mathbb{S}_4 E^* \otimes \bigwedge^2 F^* \otimes A_1 \otimes A_1 \\
\big\downarrow \\
\mathbb{S}_4 E^* \otimes A_2
\end{array}$$

and the last map is given by symmetric multiplication. The block $(9,3;4^6) \to (8,1;3^6)$ was constructed as follows:

$$\begin{array}{c}
\mathbb{S}_6 E^* \otimes \bigwedge^2 E^* \otimes \bigwedge^2 E^* \otimes \bigwedge^6 F^* \\
\big\downarrow {\scriptstyle \Delta\otimes\Delta\otimes\Delta\otimes\Delta} \\
\mathbb{S}_5 E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \otimes E^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \otimes \bigwedge^2 F^* \\
\big\downarrow {\scriptstyle m_{1,3,5}\otimes m_{2,7}\otimes m_{4,8}\otimes m_{6,9}} \\
\mathbb{S}_7 E^* \otimes A_1 \otimes A_1 \otimes A_1 \\
\big\downarrow \\
\mathbb{S}_7 E^* \otimes A_3
\end{array}$$

and the last map is given by symmetric multiplication.

The Betti table for the resolution of the normalization is

	0	1	2	3	4	5	6
total:	16	72	182	280	230	83	7
0:	1	.	.	.	.	.	.
1:	.	.	.	.	.	.	.
2:	15	72	63	.	.	.	.
3:	.	.	84	175	90	.	.
4:	.	.	35	42	.	8	.
5:	.	.	.	63	140	75	.
6:	.	.	.	.	.	.	7

Dropping the row of degree 3 in the differential d_1, we obtain a map $(2,1;1^6)\oplus(2,1;2,1^4,0) \to (1,1;1^4,0^2)$, which is a presentation for the cokernel $C(9)$ of the inclusion $\mathbb{C}[\overline{\mathcal{O}}_9] \hookrightarrow \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}}_9)]$. The resolution of $C(9)$ could only be calculated up to the second syzygies. The equivariant form of this partial resolution is:

$$(1,1;1^4,0^2) \leftarrow (2,1;1^6) \oplus (2,1;2,1^4,0) \leftarrow (4,1;2^5,0) \oplus (3,1;3,1^5) \oplus (2,2;2^2,1^4) \oplus (3,3;3^2,2^3,0)$$

and it has Betti table

	0	1	2
total:	15	72	442
2:	15	72	78
3:	.	.	84
4:	.	.	280

By constructing a partial cone by hand, we can observe that the defining ideal of $\overline{\mathcal{O}}_9$ is generated by the representation $(2,2;2^2,1^4) \oplus (3,3;3^2,2^3,0)$, corresponding to 15 equations of degree 4 and 280 equations of degree 6.

Following remark 4.2.1, we find that $\overline{\mathcal{O}}_9 = \mathcal{O}_9 \cup \mathcal{O}_7 \cup \ldots \cup \mathcal{O}_0$.

Appendix A

Equivariant maps

In this appendix, we give a brief description of the equivariant maps between representations that are used to construct the differentials in our complexes (more details can be found in [20, Ch. 1]). The notation introduced here is used extensively throughout chapters 6, 7, 8 and 9. We use the symbol $\mathbb{S}_\lambda$ to denote the Schur functor associated to the partition λ. In particular, $\mathbb{S}_i = \mathbb{S}_{(i)} = \mathrm{Sym}_i$ denotes the i-th symmetric power. In order to simplify the notation as much as possible, we omit to write any symbol for the identity and other obvious maps.

Throughout this appendix, E denotes a complex vector space of finite dimension n with basis $\{e_1, \ldots, e_n\}$; E^* denotes the dual of E and we take $\{e_1^*, \ldots, e_n^*\}$ to be the basis dual to $\{e_1, \ldots, e_n\}$. All the maps we describe are $\mathrm{GL}(E)$-equivariant.

A.1 Diagonals

Let r, s be natural numbers such that $0 \leqslant r + s \leqslant n$. The *exterior diagonal* is the map:

$$\Delta : \bigwedge^{r+s} E \longrightarrow \bigwedge^{r} E \otimes \bigwedge^{s} E$$

with

$$\Delta(e_1 \wedge \ldots \wedge e_{r+s}) = \sum_{\sigma \in \mathfrak{S}_{r+s}^{r,s}} (-1)^{\mathrm{sgn}(\sigma)} e_{\sigma(1)} \wedge \ldots \wedge e_{\sigma(r)} \otimes e_{\sigma(r+1)} \wedge \ldots \wedge e_{\sigma(r+s)}.$$

Here

$$\mathfrak{S}_{r+s}^{r,s} := \{\sigma \in \mathfrak{S}_{r+s} \mid \sigma(1) < \ldots < \sigma(r), \sigma(r+1) < \ldots < \sigma(r+s)\}$$

where $\mathfrak{S}_d$ denotes the symmetric group on d letters and $\mathrm{sgn}(\sigma)$ is the sign of the permutation σ.

Now let r, s be arbitrary natural numbers. The *symmetric diagonal* is the map:

$$\Delta : \mathbb{S}_{r+s} E \longrightarrow \mathbb{S}_r E \otimes \mathbb{S}_s E$$

with

$$\Delta(e_1 \ldots e_{r+s}) = \sum_{\sigma \in \mathfrak{S}_{r+s}^{r,s}} e_{\sigma(1)} \ldots e_{\sigma(r)} \otimes e_{\sigma(r+1)} \ldots e_{\sigma(r+s)}.$$

Both diagonals can be generalized to the case where the codomain is a tensor product of more than two exterior or symmetric powers of E in the obvious way. It will be clear from the context whether we are using the exterior or symmetric diagonal and how many and what factors we are taking in the codomain.

A.2 Multiplications

Let r, s be natural numbers such that $0 \leqslant r + s \leqslant n$. The *exterior multiplication* is the map:

$$m : \bigwedge^r E \otimes \bigwedge^s E \longrightarrow \bigwedge^{r+s} E$$

with

$$m(u_1 \wedge \ldots \wedge u_r \otimes v_1 \wedge \ldots \wedge v_s) = u_1 \wedge \ldots \wedge u_r \wedge v_1 \wedge \ldots \wedge v_s.$$

Now let r, s be arbitrary natural numbers. The *symmetric multiplication* is the map:

$$m : \mathbb{S}_r E \otimes \mathbb{S}_s E \longrightarrow \mathbb{S}_{r+s} E$$

with

$$m(u_1 \dots u_r \otimes v_1 \dots v_s) = u_1 \dots u_r v_1 \dots v_s.$$

Both multiplications can be generalized to the case where the domain is a tensor product of more than two exterior or symmetric powers of E in the obvious way. It will be clear from the context whether we are using the exterior or symmetric multiplication and how many and what factors we are taking in the domain.

When the tensor factors that we wish to multiply are not adjacent, we use subscripts to clarify which factors we are multiplying. For example, we write $m_{1,3} : E \otimes E \otimes E \rightarrow \mathbb{S}_2 E \otimes E$ to indicate we apply the symmetric multiplication to the first and third factor, leaving the second one alone.

A.3 Traces

Let r be a natural number such that $0 \leqslant r \leqslant n$. The *exterior trace* is the map:

$$tr^{(r)} : \mathbb{C} \longrightarrow \bigwedge^r E \otimes \bigwedge^r E^*$$

with

$$tr^{(r)}(1) = \sum_{1 \leqslant i_1 < \dots < i_r \leqslant n} e_{i_1} \wedge \dots \wedge e_{i_r} \otimes e_{i_1}^* \wedge \dots \wedge e_{i_r}^*.$$

Now let r be an arbitrary natural number. The *symmetric trace* is the map:

$$tr^{(r)} : \mathbb{C} \longrightarrow \mathbb{S}_r E \otimes \mathbb{S}_r E^*$$

with

$$tr^{(r)}(1) = \sum_{1 \leqslant i_1 \leqslant \ldots \leqslant i_r \leqslant n} e_{i_1} \ldots e_{i_r} \otimes e^*_{i_1} \ldots e^*_{i_r}.$$

It will be clear from the context whether we are using the exterior or symmetric trace.

A.4 Exterior duality

Let r be a natural number such that $0 \leqslant r \leqslant n$. The *exterior duality* is the map:

$$* : \bigwedge^{r} E \longrightarrow \bigwedge^{n-r} E^*$$

with

$$*(e_{i_1} \wedge \ldots \wedge e_{i_r}) = (-1)^{\mathrm{sgn}(\sigma)} e^*_{j_1} \wedge \ldots \wedge e^*_{j_{n-r}},$$

where $1 \leqslant i_1 < \ldots < i_r \leqslant n$, $1 \leqslant j_1 < \ldots < j_{n-r} \leqslant n$, $\{i_1, \ldots, i_r\} \cup \{j_1, \ldots, j_{n-r}\} = \{1, \ldots, n\}$ and σ is the permutation

$$\begin{pmatrix} 1 & \ldots & r & r+1 & \ldots & n \\ i_1 & \ldots & i_r & j_1 & \ldots & j_{n-r} \end{pmatrix}.$$

Similarly we can define an exterior duality $* : \bigwedge^r E^* \longrightarrow \bigwedge^{n-r} E$.

Appendix B

Equivariant resolutions in M2

Let G be a semisimple Lie group and V a representation of G. Consider the polynomial ring $A = \mathrm{Sym}(V)$ and a G-equivariant map of graded free A-modules:

$$\varphi : E \otimes A(-k) \longrightarrow F \otimes A(-k+d),$$

where E and F are representations of G.

Suppose we can write the matrix of φ with respect to bases of weight vectors $\{e_1, \ldots, e_n\}$ of E and $\{f_1, \ldots, f_m\}$ of F. The entry in row i and column j is the polynomial $p_{i,j} \in A_d$ which appears as the coefficient of f_i in $\varphi(e_j)$. It is easy to verify that $p_{i,j}$ is a weight vector in A_d.

Now assume that A is generated (as a ring) by the variables $x_1, \ldots, x_t$, which are weight vectors in V. This is not at all restrictive since, given any set of variables, there is always a linear change of variables to a new set which is composed of weight vectors of V. Let w_u be the weight of x_u.

If $m = x_1^{\alpha_1} \ldots x_t^{\alpha_t}$ is any monomial in $p_{i,j}$, then the weight of $p_{i,j}$ can be easily calculated using the formula:

$$\mathrm{weight}(p_{i,j}) = \sum_{u=1}^{t} \alpha_u w_u.$$

Moreover, since φ is G-equivariant, we have

$$\text{weight}(f_i) + \text{weight}(p_{i,j}) = \text{weight}(e_j).$$

In particular, if we know the weight of f_i we can recover the weight of e_j and vice versa. This can be used to devise an algorithm that will take as input the weights of the weight vectors $f_1, \dots, f_m$ and will generate as output the weights of the weight vectors $e_1, \dots, e_n$. This would allow one to recover the character of the representation E.

In this work, we are interested in minimal free resolutions that are equivariant for the action of some complex reductive group. Each differential in such a resolution is constructed from blocks like the map φ above. The representation theoretic structure of the resolution can be described using the weights for the action of the semisimple part of the group. Using the algorithm outlined above one can recover the entire representation theoretic structure, if the weights in a basis of weight vectors are known in a given homological dimension. This is further complicated by the fact that the differentials in our resolutions are not necessarily expressed in bases of weight vectors, since they are calculated using Gröbner basis methods in the software system Macaulay2 [5]. The procedure outlined above can be refined to work around this issue, as long as the columns in the matrix of each differential have different leading terms, which can easily be attained for matrices of differentials in the resolutions computed using Macaulay2. Then the leading terms carry the same weight information as the monomials in the entries $p_{i,j}$ for bases of weight vectors and can be used to recover the weights. The resulting algorithm was used while preparing this manuscript to obtain the equivariant form of resolutions of non normal orbit closures, although a complete description of the algorithm is beyond the scope of this work. We plan to release the details of this algorithm in future work alongside an implementation in a package for Macaulay2.

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