Grassmannians and Cluster Algebras

Federico Galetto

December 16, 2009

Grassmannians and Cluster Algebras

The coordinate ring of the Grassmannian $G(2, n+3)$ is the Ptolemy algebra $\mathcal{A}_n$.
Clusters in $\mathcal{A}_n$ are parametrized by triangulations of a regular $(n+3)$-gon $\mathbb{P}$. In particular,

- frozen variables correspond to the sides of $\mathbb{P}$;
- cluster variables correspond to the diagonals in a given triangulation of $\mathbb{P}$;
- mutations are given by diagonal flips;
- exchange relations are given by the short Plücker relations.

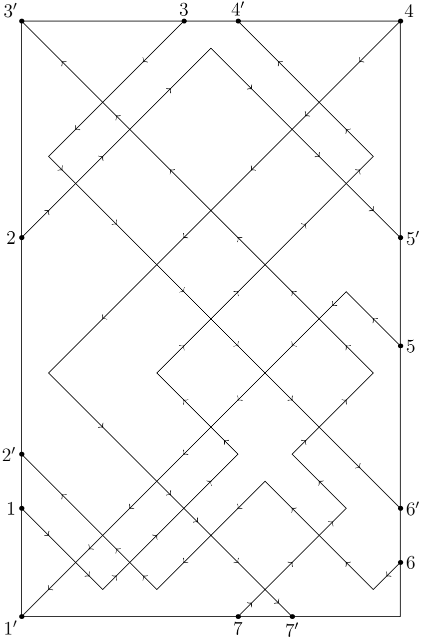
In "Grassmannians and Cluster Algebras", J. Scott proved the coordinate ring of any Grassmannian $G(k, n)$ is a cluster algebra.

Contructing a π-diagram

- In S_7, consider the permutation

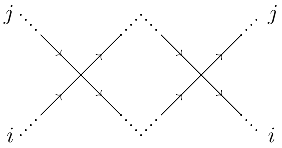
$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{pmatrix}$$

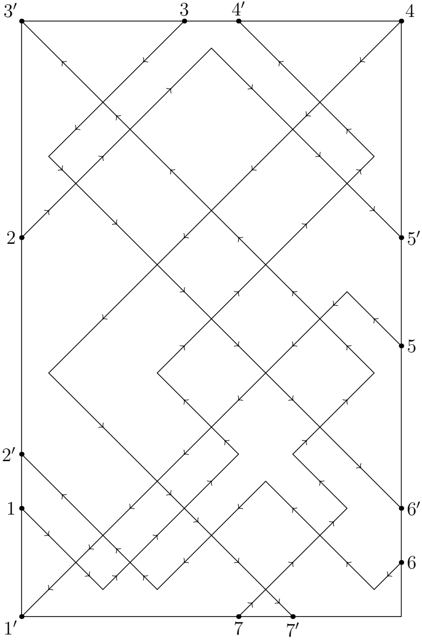
- Take a convex polygon with 14 vertices and label them $1', 1, 2', 2, \ldots, 7', 7$ clockwise.
- For i from 1 to 7, draw a path in the interior of the polygon joining i to $\pi(i)'$ and oriented towards $\pi(i)'$.



Compatibility relations

1. No path intersects itself.
2. All path intersections are transversal.
3. As a path is traversed from source to target, the paths intersecting it must alternate in orientation cutting it first right, then left, right, ..., finally left.
4. For any two paths i and j, the following configuration is forbidden





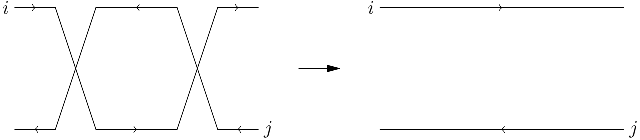
Postnikov arrangements

Definition

Let $\pi \in S_n$. Label the vertices of a convex $2n$-gon clockwise by the indices $1', 1, 2', 2, \ldots, n', n$. A *Postnikov arrangement* for π (or a π-*diagram*) is a collection of n oriented paths in the interior of the polygon; the i-th path joins the vertex i with the vertex $\pi(i)'$ and is directed towards $\pi(i)'$. The collection of paths must satisfy the compatibility relations $1 - 4$.

Postnikov arrangements are identified up to:

- *isotopy*, i.e. distortions of the configuration that neither introduce nor remove crossings;
- *untwisting* conescutive crossings of two paths



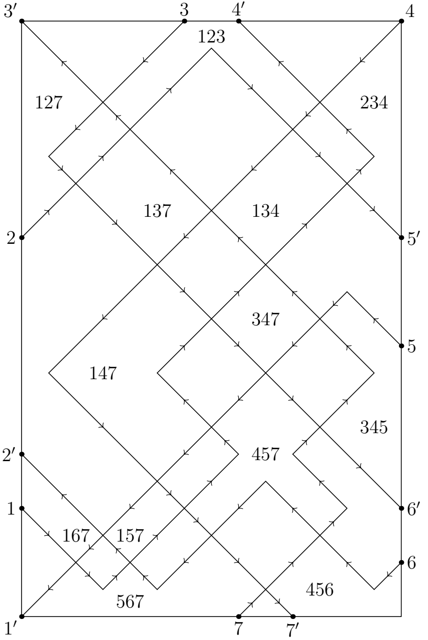
Labels in a π-diagram

A region in a π-diagram is called:

- *odd*, if its boundary is oriented (either clockwise or counterclockwise);
- *even*, if its boundary paths alternate in orientation, i.e. if the boundary of the region changes orientation at every intersection with another path.

The boundary of the $2n$-gon is oriented clockwise.

Label an even region with the index i if the circuit obtained traversing the i-th wire and then the boundary of the polygon clockwise from $\pi(i)'$ to i does not wind around the region.



The Grassmann permutation

Let k, n be integers with $0 < k < n$. We call

$$\pi_{k,n} = \begin{pmatrix} 1 & \dots & n-k & n-k+1 & \dots & n \\ k+1 & \dots & n & 1 & \dots & k \end{pmatrix}$$

the *Grassmann permutation*.

Proposition (Postnikov)

Let $\pi_{k,n}$ be the Grassmann permutation.

1. The number of even regions in a $\pi_{k,n}$-diagram is $k(n-k)+1$.
2. Each even region is labeled by exactly k indices from $[1 \dots n]$.
3. The k-subsets labeling boundary cells are always the intervals

$$[1 \dots k],\ [2 \dots k+1],\ [3 \dots k+2],\ \dots,\ [n \dots k-1].$$

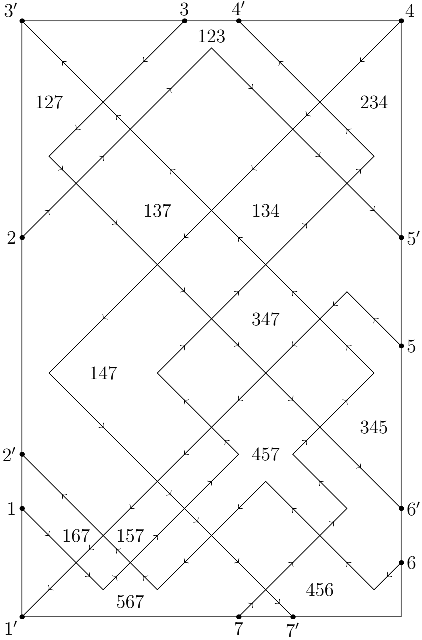
4. Every k-subset in $[1 \dots n]$ occurs as the labeling set of an even cell in some $\pi_{k,n}$-diagram.

Example: a $\pi_{3,7}$-diagram

- The permutation

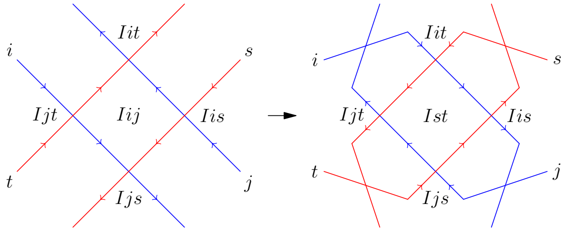
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \end{pmatrix}$$

  is the Grassmann permutation $\pi_{3,7}$
- There are 13 even cells.
- Each even cell is labeled by three distinct indices from $[1 \dots 7]$.
- There are 7 boundary cells labeled by intervals in $[1 \dots 7]$.



Geometric exchange

Given a π-diagram $\mathbf{A}$ and an even quadrilateral cell inside $\mathbf{A}$, ($|I| = k - 2$ and i, j, s, t are distinct indices disjoint from I)



a new π-diagram is constructed by the above local rearrangement, called a *geometric exchange*.

The geometric exchange is an involution, provided we untwist consecutive crossings after performing the exchange.

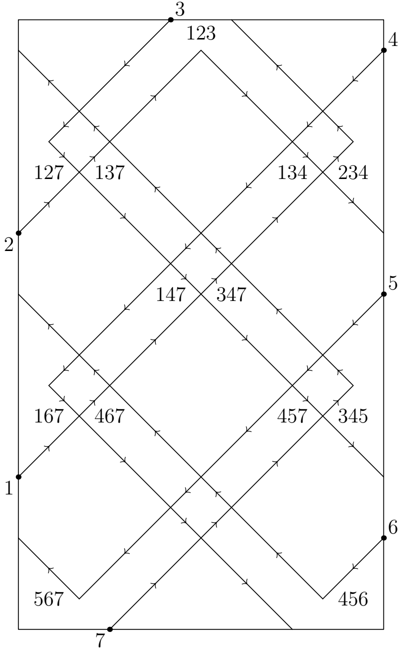
Proposition (Postnikov)

Let $\mathbf{A}$ and $\mathbf{A}'$ be two $\pi_{k,n}$-diagrams. Then there is a sequence of geometric exchanges transforming $\mathbf{A}$ into $\mathbf{A}'$.

Quadrilateral Postnikov Arrangements

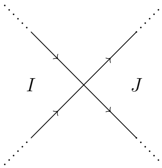
For positive integers k and n with $n \geqslant k+2 \geqslant 4$ there exists a $\pi_{k,n}$-diagram, denoted as $\mathbf{A}_{k,n}$, whose internal even cells are all quadrilateral.

These arrangements are necessary for the proof of the main result.



The matrix of a $\pi_{k,n}$-diagram

Two even regions of a Postnikov arrangement, with labels I and J, are said to be *neighbors* if locally they are situated as follows

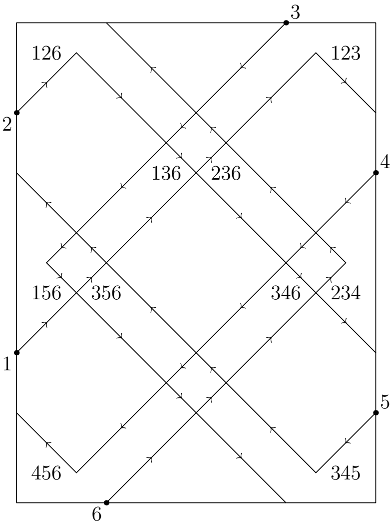


I is said to be *oriented towards* J and J is said to be *oriented away* from I. Given a $\pi_{k,n}$-diagram $\mathbf{A}$, let $\tilde{\mathbf{B}}(\mathbf{A})$ be the integer matrix with rows indexed by the k-subset labels of $\mathbf{A}$, columns indexed by the interior k-subset labels of $\mathbf{A}$ and entries

$$b_{I,J} = \begin{cases} 1, & \text{if } I \text{ is oriented towards } J, \\ -1, & \text{if } I \text{ is oriented away from } J, \\ 0, & \text{otherwise.} \end{cases}$$

The principal submatrix $\mathbf{B}(\mathbf{A})$ is clearly skew-symmetric.

Example: $\tilde{\mathbf{B}}(\mathbf{A}_{3,6})$



	136	236	346	356
136	0	1	0	-1
236	-1	0	1	0
346	0	-1	0	1
356	1	0	-1	0
123	0	-1	0	0
234	0	1	-1	0
345	0	0	1	0
456	0	0	0	-1
156	-1	0	0	1
126	1	0	0	0

The cluster algebra $\mathcal{A}_{k,n}$

Let $\mathcal{F}$ be the field of rational functions generated by the indeterminates $[K]$ for k-subset labels K arising in the quadrilateral Postnikov arrangement $\mathbf{A}_{k,n}$.

Let $\mathcal{A}_{k,n}$ denote the cluster algebra generated inside $\mathcal{F}$ by the initial seed $(\mathbf{x}(\mathbf{A}_{k,n}), \tilde{\mathbf{B}}(\mathbf{A}_{k,n}))$, where

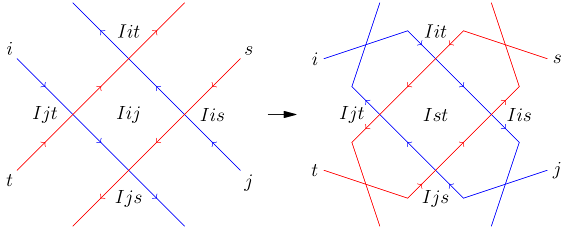
$$\mathbf{x}(\mathbf{A}_{k,n}) = \{[K] \mid K \text{ interior } k\text{-subset label in } \mathbf{A}_{k,n}\}$$

and $\tilde{\mathbf{B}}(\mathbf{A}_{k,n})$ is the matrix defined earlier.

If $K_1, \ldots, K_n$ are the boundary k-subset labels in $\mathbf{A}_{k,n}$, then $\mathcal{A}_{k,n}$ is an algebra over the ring $\mathbb{C}[[K_1], \ldots, [K_n]]$.

Exchange relations in $\mathcal{A}_{k,n}$

Let $|I| = k-2$ and i, j, s, t are distinct indices disjoint from I.



Theorem (Scott)

Each $\pi_{k,n}$-diagram $\mathbf{A}$ gives rise to a seed $(\mathbf{x}(\mathbf{A}), \tilde{\mathbf{B}}(\mathbf{A}))$ in $\mathcal{A}_{k,n}$ whose cluster variables are indexed by the interior k-subset labels of $\mathbf{A}$ and with the property that if $\mathbf{A}'$ is obtained from $\mathbf{A}$ by a single geometric exchange through a quadrilateral cell labeled K in $\mathbf{A}$, then $\mu_K(\tilde{\mathbf{B}}(\mathbf{A})) = \tilde{\mathbf{B}}(\mathbf{A}')$.

The exchange relation corresponding to the geometric exchange above is

$$[Ist][Iij] = [Iit][Ijs] + [Ijt][Iis].$$

$\mathcal{A}_{k,n}$ and $\mathbb{C}[G(k,n)]$

Every k-subset K of $[1 \dots n]$ identifies a Plücker coordinate Δ^K in $\mathbb{C}[G(k,n)]$.

In particular, identifying $[K_1], \dots, [K_n]$ with the Plücker coordinates $\Delta^{K_1}, \dots, \Delta^{K_n}$, we deduce that $\mathbb{C}[G(k,n)]$ has a natural structure of algebra over $\mathbb{C}[[K_1], \dots, [K_n]]$.

Theorem (Scott)

There is an isomorphism $\mathcal{A}_{k,n} \to \mathbb{C}[G(k,n)]$ of $\mathbb{C}[[K_1], \dots, [K_n]]$-algebras sending $[K]$ to Δ^K for every k-subset K of $[1 \dots n]$.

The proof is based on the 'geometric realization criterion' proved by Fomin and Zelevinsky in "Cluster Algebra II".

Grassmannians of finite type

Let $n \geqslant 3$. $\mathbb{C}[G(2,n)]$ is a cluster algebra of finite type: it has finitely many seeds, corresponding to triangulations of a regular n-gon.

The cluster variables are the Plücker coordinates Δ^{ij} for $1 \leqslant i < j \leqslant n$.

The Plücker coordinates are cluster variables in the coordinate ring of $G(k,n)$. However, in general, there will be more cluster variables than Plücker coordinates.

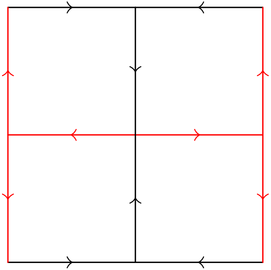
Theorem (Scott)

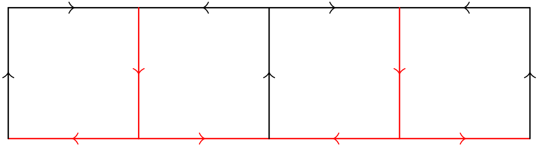
$G(3,6)$, $G(3,7)$ and $G(3,8)$ are the only Grassmannians $G(k,n)$, within the range $2 < k \leqslant \frac{n}{2}$, whose coordinate rings are cluster algebras of finite type.

To determine finiteness, we analyze the graph $\Gamma(\mathbf{B}(\mathbf{A}_{k,n}))$, where $\mathbf{A}_{k,n}$ is the quadrilateral $\pi_{k,n}$-diagram associated to the initial seed of $\mathcal{A}_{k,n}$.

The infinite case

Let $(k, n) \neq (3,6), (3,7), (3,8)$. Then $\Gamma(\mathbf{B}(\mathbf{A}_{k,n}))$ contains, as an induced subgraph, one of the following:

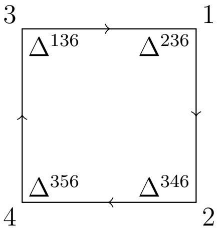




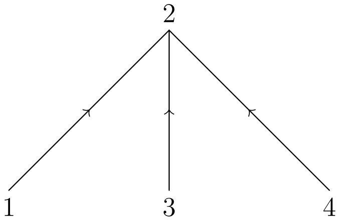
These two subgraphs contain, as induced subgraphs, a copy of the extended Dynkin diagram $D_6^{(1)}$.

$\mathbb{C}[G(3,6)]$

The graph $\Gamma(\mathbf{B}(\mathbf{A}_{3,6}))$ is



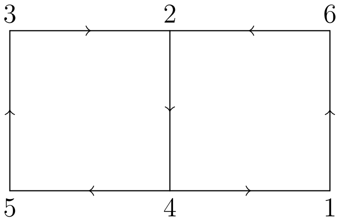
By performing a sequence of mutations at $4, 2, 4, 1$, we obtain



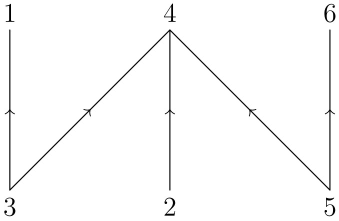
which is the Dynkin diagram D_4.

$\mathbb{C}[G(3,7)]$

The graph $\Gamma(\mathbf{B}(\mathbf{A}_{3,7}))$ is



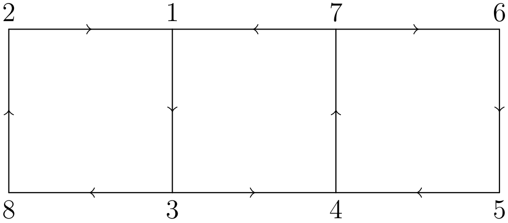
Through the sequence of mutations at the vertices $2, 4, 3, 5, 6, 5, 1$, we obtain



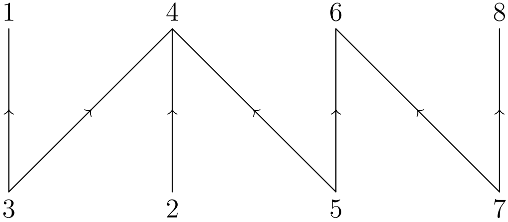
which is the Dynkin diagram E_6.

$\mathbb{C}[G(3,8)]$

The graph $\Gamma(\mathbf{B}(\mathbf{A}_{3,8}))$ is



Mutating at the nodes $1, 3, 7, 6, 5, 2, 4, 3, 8, 7, 6$, we obtain



which is the Dynkin diagram E_8.

Projective plane geometry and cluster variables

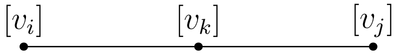
Since $\mathbb{C}[G(3,n)]$, for $n = 6, 7, 8$, is a cluster algebra of finite type, we would like to describe all its cluster variables. We already know some of them are the Plücker coordinates.

As an element of $\mathbb{C}[G(3,n)]$, a cluster variable is a regular function on $G(3,n)$, so we can describe its vanishing locus.

A point in $G(3,n)$ is a 3-dimensional vector subspace of $\mathbb{C}^n$. As such, it is identified by a $3 \times n$ matrix over $\mathbb{C}$. The columns of this matrix define n vectors $v_1, \ldots, v_n \in \mathbb{C}^3$.

If we restrict ourselves to the open subset of $G(3,n)$ where none of the v_i vanish, then we get a configuration of n points $[v_1], \ldots, [v_n] \in \mathbb{CP}^2$.

With this setup, the Plücker coordinate Δ^{ijk} vanishes on those points of $G(3,n)$ for which $[v_i], [v_j]$ and $[v_k]$ in the associated configuration are colinear.

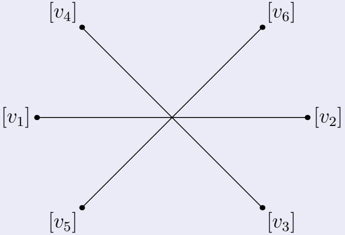


Cluster variables in $\mathbb{C}[G(3,6)]$

Theorem (Scott)

$\mathbb{C}[G(3,6)]$ possesses 16 cluster variables:

- 14 Plücker coordinates Δ^{ijk}, with $\{i,j,k\}$ an internal 3-subset of $[1\dots 6]$;
- X^{123456} a quadratic regular function which vanishes on configurations of points of the type:



- $Y^{123456}(v_1,v_2,v_3,v_4,v_5,v_6) = X^{123456}(v_6,v_1,v_2,v_3,v_4,v_5)$, another quadratic regular function having the same type of vanishing locus as X^{123456} but with the indices cyclically shifted.

Cluster variables in $\mathbb{C}[G(3,7)]$

Assume $n > 6$ and $I \subset [1 \ldots n]$ with $|[1 \ldots n] - I| = 6$.
Let $I : G(3,n) \to G(3,6)$ be the projection that takes the $3 \times n$ matrix representing a point in $G(3,n)$ and drops the columns indexed by I.
For $I = \{i\} \subset [1 \ldots 7]$, define

$$X^{[1\ldots 7]-\{i\}} := X^{123456} \circ I,$$
$$Y^{[1\ldots 7]-\{i\}} := Y^{123456} \circ I.$$

Theorem (Scott)

$\mathbb{C}[G(3,7)]$ possesses 42 cluster variables:

- 28 Plücker coordinates Δ^{ijk}, with $\{i,j,k\}$ an internal 3-subset of $[1 \ldots 7]$;
- 14 quadratic regular functions $X^{[1\ldots 7]-\{i\}}$ and $Y^{[1\ldots 7]-\{i\}}$ defined above for $i \in [1 \ldots 7]$.

Cluster variables in $\mathbb{C}[G(3,8)]$

The dihedral group D_n acts on a $3 \times n$ matrix representing a point of the Grassmannian $G(3,n)$ by permuting the columns of the matrix. Equivalently it acts by permuting points of the configuration $[v_1], \ldots, [v_n] \in \mathbb{CP}^2$.

If $x \in \mathbb{C}[G(3,n)]$ is a cluster variable, then, for any $\rho \in D_n$, $x \circ \rho$ is, up to sign, another cluster variable, called a *dihedral translate* of x.

Theorem (Scott)

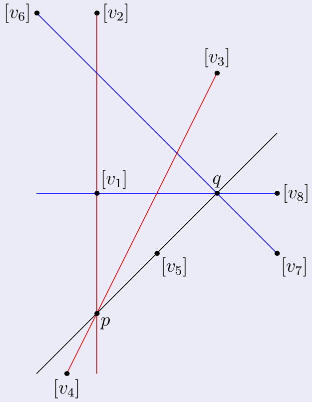
$\mathbb{C}[G(3,8)]$ possesses 128 cluster variables:

- 48 Plücker coordinates Δ^{ijk}, with $\{i,j,k\}$ an internal 3-subset of $[1\ldots 8]$;
- 56 quadratic regular functions $X^{[1\ldots 8]-\{ij\}}$ and $Y^{[1\ldots 8]-\{ij\}}$ defined above for $1 \leqslant i < j \leqslant 8$.

Cluster variables in $\mathbb{C}[G(3,8)]$

Theorem (Scott)

- A a cubic regular function which vanishes on configurations of points of the type:



- $B(v_1, v_2, v_3, \ldots, v_8) = A(v_2, v_1, v_3, \ldots, v_8)$;
- 22 dihedral translates of A and B.