An introduction to Hodge algebras

Federico Galetto

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The Grassmannian

Let k be a field, E a k-vector space of dimension m. Define

$$\mathrm{Grass}(n, E) := \{V \subseteq E \mid \dim V = n\}.$$

If $\{e_1, \ldots, e_m\}$ is a basis of E and $\{v_1, \ldots, v_n\}$ is a basis of V, then

$$\begin{aligned} v_1 &= a_{11}e_1 + \ldots + a_{1m}e_m, \\ &\vdots \\ v_n &= a_{n1}e_1 + \ldots + a_{nm}e_m. \end{aligned}$$

Let I be any sequence of indeces $1 \leqslant i_1 < \ldots < i_n \leqslant m$ and denote by p_I the $n \times n$ minor of the matrix (a_{ji}) corresponding to the columns $i_1, \ldots, i_n$.

- $\dim V = n \Rightarrow \exists I$ such that $p_I \neq 0$.
- Changing basis of V, we obtain the same p_I up to a scalar multiple.

The p_I are called *Plücker coordinates* and determine a point in $\mathbb{P}^{\binom{m}{n}-1}$.

The Plücker relations

Suppose we have $[p_I] \in \mathbb{P}^{\binom{m}{n}-1}$. Does it determine a subspace of E?
It does if and only if its coordinates satisfy some homogeneous equations called *Plücker relations*.
For example, suppose $E = k^5$ and $n = 2$. The Plücker coordinates are the minors of

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \end{pmatrix}$$

Here we have the following Plücker relation:

$$\begin{aligned} &(a_{11}a_{24} - a_{21}a_{14})(a_{12}a_{23} - a_{22}a_{13}) \\ -&(a_{11}a_{23} - a_{21}a_{13})(a_{12}a_{24} - a_{22}a_{14}) \\ +&(a_{11}a_{22} - a_{21}a_{12})(a_{13}a_{24} - a_{23}a_{14}) = 0. \end{aligned}$$

In terms of minors, the equation becomes:

$$p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34} = 0.$$

The coordinate ring of Grass(n, k^m)

Let $X = (X_{ij})$ be an $n \times m$ matrix of indeterminates over a field k $(n \leqslant m)$. Let $G_{n,m}$ be the subring of $k[X_{ij}]$ generated by all $n \times n$ minors of X. $G_{n,m}$ is the homogeneous coordinate ring of Grass(n, k^m).

Let the symbol $\boxed{\begin{array}{c|c|c} i_1 & \ldots & i_n \end{array}}$ denote the minor of X corresponding to the columns $i_1, \ldots, i_n$. Notice that the symbol $\boxed{\begin{array}{c|c|c} i_1 & \ldots & i_n \end{array}}$ is alternating in the indices $i_1, \ldots, i_n$.

$G_{n,m}$ is generated as a k-algebra by the set

$$H = \left\{ \boxed{\begin{array}{c|c|c} i_1 & \ldots & i_n \end{array}} \mid 1 \leqslant i_1 < \ldots < i_n \leqslant m \right\}.$$

If we regard the symbols of H as letters, we can say that an element in $G_{n,m}$ is a k-linear combination of monomials in those letters.

$G_{2,5}$: our running example

$G_{2,5}$ is generated by 2×2 minors of

$$\begin{pmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} \end{pmatrix}$$

Hence it is a k-algebra generated by the set

$$H = \Big\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}, \\ \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \Big\}.$$

We use the following notation for the product of two minors:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} := \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array} \cdot \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array} = (X_{11}X_{24} - X_{21}X_{14})(X_{12}X_{23} - X_{22}X_{13}).$$

With this notation, the Plücker relation $p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34} = 0$ becomes:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} = 0.$$

Standard monomials in $G_{n,m}$

To identify a monomial in the letters of H, we use the tableaux

i_1	$\dots$	i_n
j_1	$\dots$	j_n
$\vdots$	$\vdots$	$\vdots$
u_1	$\dots$	u_n

We call this a *standard* monomial if its rows are strictly increasing and its columns are weakly increasing.
Notice that if we multiply a standard monomial with a nonstandard one, we get a nonstandard monomial.

Claim: the standard monomials generate $G_{n,m}$ as a k-vector space.

To prove the claim, it is enough to show that a nonstandard product of two minors is a k-linear combination of standard monomials.

$G_{2,5}$: the straightening relations

Recall that the following Plücker relation

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} + \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} = 0$$

holds in $G_{2,5}$.

The monomial $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}$ is nonstandard, while $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ are both standard.

Hence we get the equation:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array},$$

which is called *straightening relation* for the nonstandard monomial $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}$.

$G_{2,5}$: the straightening relations

Here is a list of all nonstandard products of two minors in $G_{2,5}$ together with their straightening relations:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 5 \\ \hline \end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline \end{array} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline \end{array} - \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline \end{array}$$

$G_{2,5}$: ordering the minors

Let $\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline\end{array}$, $\begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline\end{array} \in H$. Set

$$\begin{array}{|c|c|}\hline i_1 & i_2 \\ \hline\end{array} \leqslant \begin{array}{|c|c|}\hline j_1 & j_2 \\ \hline\end{array} \iff i_1 \leqslant j_1, i_2 \leqslant j_2.$$

$\leqslant$ is a partial order on H.
It is not total: for example, $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$ are not comparable.
Now let us look at one straightening relation:

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}.$$

Notice that $\begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}$ divides the l.h.s., while on the r.h.s. we have that:

- $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array}$ is divisible by $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array} < \begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}$,
- $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array}$ is divisible by $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array} < \begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}$.

A similarly property holds for $\begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}$ and for the other straightening relations (I will refer to it as the "H2" condition).

$G_{2,5}$: properties so far

$G_{2,5}$, the homogeneous coordinate ring of Grass$(2, k^5)$:

- is a commutative k-algebra generated by

$$H = \{\boxed{1 \mid 2}, \boxed{1 \mid 3}, \boxed{1 \mid 4}, \boxed{1 \mid 5}, \boxed{2 \mid 3}, \\ \boxed{2 \mid 4}, \boxed{2 \mid 5}, \boxed{3 \mid 4}, \boxed{3 \mid 5}, \boxed{4 \mid 5}\};$$

- is generated as a k-vector space by the standard monomials;
- is endowed with straightening relations that enable us to express the nonstandard monomials as k-linear combinations of the standard ones;
- comes with a partial order on the set of generators H that satisfies the H2 condition.

Monomials

Let H be a finite set.

Definition

A monomial on H is an element of $\mathbb{N}^H$, i.e. a function $M : H \to \mathbb{N}$.

If we think of H as the set of indeterminates in a polynomial ring $k[H]$, then we can associate to M a monomial in the usual sense, namely

$$\prod_{x \in H} x^{M(x)}.$$

Given two monomials $M, N \in \mathbb{N}^H$, their product is defined by:

$$(MN)(x) := M(x) + N(x).$$

We say that N divides M if $N(x) \leqslant M(x)$ for every $x \in H$.

Ideals of monomials

Definition

An *ideal of monomials* on H is a subset $\Sigma \subseteq \mathbb{N}^H$ such that

$$M \in \Sigma,\ N \in \mathbb{N}^H \Rightarrow MN \in \Sigma.$$

Definition

A monomial M is called *standard* with respect to the ideal Σ if $M \notin \Sigma$.

Definition

A *generator* of an ideal Σ is an element of Σ which is not divisible by any other element of Σ.

The set of generators of an ideal Σ is finite.

Hodge algebra

Consider

- R commutative ring;
- A commutative R-algebra;
- $H \subseteq A$ finite partially ordered set;
- Σ ideal of monomials on H.

To each monomial M on H, we can associate an element in A that we still denote by M:

$$M := \prod_{x \in H} x^{M(x)}.$$

Hodge algebra

Definition

A is a *Hodge algebra* governed by Σ and generated by H if:

H1 A is a free R-module on the standard monomials with respect to Σ

H2 if $N \in \Sigma$ is a generator and

$$N = \sum_i r_i M_i, \qquad 0 \neq r_i \in R,$$

is the unique expression for $N \in A$ as a linear combination of standard monomials (guaranteed by H1), then for each $x \in H$

$$x | N \Rightarrow \forall i \ \exists y_i \in H \text{ such that } y_i | M_i \text{ and } y_i < x$$

The relations in H2 are called the *straightening relations* of A.

$G_{2,5}$: the Hodge algebra structure

$G_{2,5}$ is a Hodge algebra over k generated by

$$H = \Big\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array} \Big\}$$

governed by the ideal of monomials

$$\Sigma = \left\langle \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline\end{array} \right\rangle$$

with straightening relations given by

$$\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 5 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 5 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 4 & 5 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 3 & 5 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 4 & 5 \\ \hline\end{array}$$

$$\begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline\end{array} = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 3 & 5 \\ \hline\end{array} - \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 4 & 5 \\ \hline\end{array}$$

Motivation

Hodge algebras arise as coordinate rings of algebraic varieties, for example

- Grassmannians;
- determinantal varieties;
- flag manifolds;
- Schubert varieties.

The discrete Hodge algebra

Definition

A Hodge algebra A is called *discrete* if the right hand side of all straightening relations is 0, i.e. if $N = 0$ in A for all $N \in \Sigma$.

If $R[H]$ is the polynomial ring over R whose indeterminates are the elements of H, then

$$R[H]/\Sigma R[H]$$

is a discrete Hodge algebra and any other discrete Hodge algebra governed by Σ is isomorphic to it.

To measure how far a Hodge algebra A is from being discrete, we introduce the *indescrete part* $\operatorname{Ind} A \subseteq H$ of A defined as

$$\{x \in H \mid x \text{ appears in the r.h.s. of the straightening relations for } A\}$$

The simplification of Hodge algebras

Consider a multiplicative filtration of A, i.e. a chain of ideals

$$\mathcal{I} : A = I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \dots$$

such that $I_p I_q \subseteq I_{p+q}$ $\forall p, q \geqslant 0$ and $R \cap I_1 = \{0\}$.
To this filtration we can associate a new R-algebra by setting

$$\operatorname{gr}_{\mathcal{I}} A := A/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \dots$$

Theorem

If $x \in H$ is a minimal element of $\operatorname{Ind} A$ *and*

$$\mathcal{I} = \{x^n A\} : A \supseteq xA \supseteq x^2 A \supseteq x^3 A \supseteq \dots,$$

then $\operatorname{gr}_{\mathcal{I}} A$ *is a Hodge algebra governed by* Σ *with*

$$\operatorname{Ind}(\operatorname{gr}_{\mathcal{I}} A) \subseteq \operatorname{Ind} A \setminus \{x\}.$$

The simplification of Hodge algebras

Corollary

If A is a Hodge algebra governed by Σ, there is a sequence of elements $x_1, \dots, x_n \in \operatorname{Ind} A$ such that defining

$$A_n := A, \qquad A_{i-1} := \operatorname{gr}_{\{x_i^n A_i\}} A_i \ \ \forall i = 1, \dots, n$$

we have:

- *each A_i is a Hodge algebra governed by Σ*
- *x_i is minimal in $\operatorname{Ind} A_i$*
- *A_0 is discrete.*

This result may be viewed as a stepwise flat deformation, whose most general fiber is A and whose most special fiber is the discrete Hodge algebra $R[H]/\Sigma R[H]$.

Properties preserved under deformation

The previous result allows us to reduce many questions about a Hodge algebra A to questions about more nearly discrete and therefore simpler Hodge algebras.
In particular, many interesting properties that are satisfied by A_0 are preserved under the deformation and are also satisfied by A.

Theorem

- *If A_0 is reduced, then A is.*
- *If A_0 is Cohen-Macaulay, then A is.*
- *If A_0 is Gorenstein, then A is.*
- *If R is a field or $\mathbb{Z}$, then* $\dim A = \dim A_0$.

Ideals generated by monomials

Recall that discrete Hodge algebras are isomorphic to $R[H]/\Sigma R[H]$, i.e. they are quotients of polynomial rings modulo ideals generated by monomials.

Proposition

Suppose R is a domain. Let Σ be an ideal of monomials and set $I = \Sigma R[H]$.

- I is prime $\iff$ Σ is generated by a subset of H.
- I is radical $\iff$ Σ is generated by square-free monomials.
- I is primary $\iff$ whenever $x \in H$ divides a generator of Σ, there is a generator which is a power of x.
- The associated primes of I are all generated by subsets H.

$G_{2,5}$: the discrete Hodge algebra

The discrete Hodge algebra A_0 obtained by deforming $G_{2,5}$, is the polynomial ring:

$$k\left[\begin{array}{|c|c|}\hline 1 & 2 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline\end{array}\right]$$

modulo the ideal

$$I = \left(\begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline\end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline\end{array}\right)$$

For example, we may notice that:

- A_0 is not a domain, since I is not generated by a subset of H;
- A_0 is reduced, since I is generated by square-free monomials.

As a consequence, we deduce that $G_{2,5}$ is reduced.

Simplicial complexes

Let H be a finite set.

Definition

We say Δ is a *simplicial complex* with vertex set H, if Δ is a collection of subsets of H (called *faces*) such that:

- $\forall x \in H$, $\{x\} \in \Delta$;
- $T \subseteq S \in \Delta \Rightarrow T \in \Delta$.

Definition

The *dimension* of a face S in Δ is defined as $|S| - 1$.

Definition

The *dimension* of Δ is the maximum of the dimensions of its faces.

Ideals of monomials and simplicial complexes

Let $S \subseteq H$ and define a monomial $\chi_S \in \mathbb{N}^H$ by

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \notin S \end{cases}$$

Suppose $\Sigma \subseteq \mathbb{N}^H$ is an ideal of monomials such that

- Σ is generated by square-free monomials;
- $\forall x \in H$, $\chi_{\{x\}} \notin \Sigma$.

If we define

$$\Delta := \{S \subseteq H \mid \chi_S \notin \Sigma\},$$

then Δ is a simplicial complex with vertex set H.

Proposition

- The minimal primes of A_0 are generated by the complements of the maximal faces of Δ.
- If R is Noetherian, then $\dim A_0 = \dim R + \dim \Delta + 1$ and height $HA_0 = \dim \Delta + 1$.

$G_{2,5}$: the dimension of $G_{2,5}$

Recall

$$\Sigma = \left\langle \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline \end{array} \right\rangle$$

If we construct Δ as before, the maximal faces are

- $\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right\}$,
- $\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right\}$,
- $\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right\}$,
- $\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right\}$,
- $\left\{ \begin{array}{|c|c|}\hline 1 & 2 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 3 & 5 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 4 & 5 \\ \hline \end{array} \right\}$.

Therefore

$$\dim A_0 = \dim k + \dim \Delta + 1 = 0 + 6 + 1 = 7.$$

As a consequence, $\dim G_{2,5} = 7$.

Wonderful posets

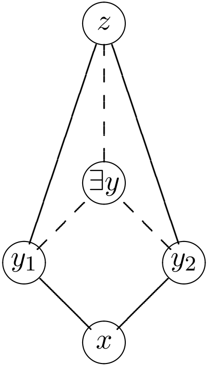
Let H be a poset.

Definition

An element $y \in H$ is a *cover* of an element $x \in H$ if $x < y$ and no element of H is properly between x and y.

Definition

H is *wonderful* if the following condition holds in the poset $H \cup \{-\infty, \infty\}$ obtained by adjoining least and greatest elements to H:
if $y_1, y_2 < z$ are covers of an element x, then there is an element $y \leqslant z$ which is a cover of both y_1 and y_2.



Some terminology for posets

Let H be a finite poset.

Definition

A *chain* in H is a totally ordered set $X \subseteq H$; its *length* is $|X| - 1$.

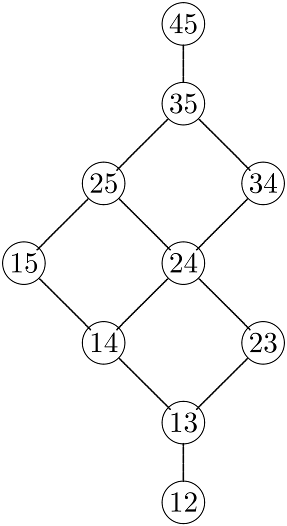
Definition

The *dimension* of H is the maximum of the lengths of chains in H.

Definition

The *height* of an element $x \in H$, denoted $\operatorname{ht} x$, is the maximum of the lengths of chains descending from x.

$G_{2,5}$: H is a wonderful poset



On the left is a diagram of the poset H for $G_{2,5}$ (smaller elements appear on the bottom).
We see that:

- H is a wonderful poset;
- $\dim H = 6$;
- the elements of height i are the ones appearing on the i-th row starting from the bottom and counting from 0.

Recall

$$\Sigma = \left\langle \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 3 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 2 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 1 & 5 \\ \hline 3 & 4 \\ \hline \end{array}, \begin{array}{|c|c|}\hline 2 & 5 \\ \hline 3 & 4 \\ \hline \end{array} \right\rangle$$

Notice that Σ is generated by the products of the pairs of elements which are incomparable in the partial order on H.

Wonderful posets and regular sequences

Let A be a Hodge algebra generated by H and governed by Σ.

Definition

A is called *ordinal* if Σ is generated by the products of the pairs of elements which are incomparable in the partial order on H.

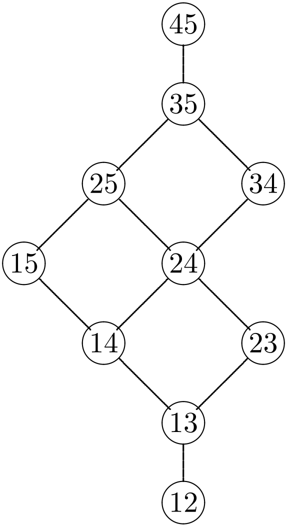
Theorem

Let A be an ordinal Hodge algebra and set

$$p_i = \sum_{x \in H : \operatorname{ht} x = i} x.$$

If H is wonderful and $\dim H = n$, *then $p_0, \ldots, p_n$ is a regular sequence.*

$G_{2,5}$: a regular sequence



Let A_0 be the discrete Hodge algebra obtained from $G_{2,5}$.
The elements:

$$p_0 = \boxed{1 \mid 2}$$
$$p_1 = \boxed{1 \mid 3}$$
$$p_2 = \boxed{1 \mid 4} + \boxed{2 \mid 3}$$
$$p_3 = \boxed{1 \mid 5} + \boxed{2 \mid 4}$$
$$p_4 = \boxed{2 \mid 5} + \boxed{3 \mid 4}$$
$$p_5 = \boxed{3 \mid 5}$$
$$p_6 = \boxed{4 \mid 5}$$

form a regular sequence in HA_0.
Since $\dim A_0 = 7$, A_0 is Cohen-Macaulay.
As a consequence, $G_{2,5}$ is Cohen-Macaulay.

References

- De Concini, C., Eisenbud, D., Procesi, C.: Hodge algebras, Astérisque 91 (1982)
- De Concini, C., Eisenbud, D., Procesi, C.: Young diagrams and determinantal varieties, Inventiones math. 56, 129–165 (1980)