An introduction to Hodge algebras

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The Grassmannian

Let \( k \) be a field, \( E \) a \( k \)-vector space of dimension \( m \). Define

\[
\text{Grass}(n, E) := \{ V \subseteq E \mid \dim V = n \}.
\]

If \( \{ e_1, \ldots, e_m \} \) is a basis of \( E \) and \( \{ v_1, \ldots, v_n \} \) is a basis of \( V \), then

\[
v_1 = a_{11} e_1 + \ldots + a_{1m} e_m, \\
v_n = a_{n1} e_1 + \ldots + a_{nm} e_m.
\]

Let \( I \) be any sequence of indeces \( 1 \leq i_1 < \ldots < i_n \leq m \) and denote by \( p_I \) the \( n \times n \) minor of the matrix \( (a_{ji}) \) corresponding to the columns \( i_1, \ldots, i_n \).

- \( \dim V = n \Rightarrow \exists I \) such that \( p_I \neq 0 \).
- Changing basis of \( V \), we obtain the same \( p_I \) up to a scalar multiple.

The \( p_I \) are called \textit{Plücker coordinates} and determine a point in \( \mathbb{P}^{m-1} \).
The Plücker relations

Suppose we have \([p_I] \in \mathbb{P}^{\binom{m}{n}-1}\). Does it determine a subspace of \(E\)? It does if and only if its coordinates satisfy some homogeneous equations called *Plücker relations*.

For example, suppose \(E = k^5\) and \(n = 2\). The Plücker coordinates are the minors of

\[
\begin{pmatrix}
 a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\
 a_{21} & a_{22} & a_{23} & a_{24} & a_{25}
\end{pmatrix}
\]

Here we have the following Plücker relation:

\[
(a_{11}a_{24} - a_{21}a_{14})(a_{12}a_{23} - a_{22}a_{13})
- (a_{11}a_{23} - a_{21}a_{13})(a_{12}a_{24} - a_{22}a_{14})
+ (a_{11}a_{22} - a_{21}a_{12})(a_{13}a_{24} - a_{23}a_{14}) = 0.
\]

In terms of minors, the equation becomes:

\[ p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34} = 0. \]
The coordinate ring of Grass($n, k^m$)

Let $X = (X_{ij})$ be an $n \times m$ matrix of indeterminates over a field $k$ ($n \leq m$). Let $G_{n,m}$ be the subring of $k[X_{ij}]$ generated by all $n \times n$ minors of $X$. $G_{n,m}$ is the homogeneous coordinate ring of Grass($n, k^m$).

Let the symbol $\begin{array}{c} i_1 \\ \vdots \\ i_n \end{array}$ denote the minor of $X$ corresponding to the columns $i_1, \ldots, i_n$. Notice that the symbol $\begin{array}{c} i_1 \\ \vdots \\ i_n \end{array}$ is alternating in the indices $i_1, \ldots, i_n$.

$G_{n,m}$ is generated as a $k$-algebra by the set

$$H = \left\{ \begin{array}{c} i_1 \\ \vdots \\ i_n \end{array} \mid 1 \leq i_1 < \ldots < i_n \leq m \right\}.$$ 

If we regard the symbols of $H$ as letters, we can say that an element in $G_{n,m}$ is a $k$-linear combination of monomials in those letters.
$G_{2,5}$: our running example

$G_{2,5}$ is generated by $2 \times 2$ minors of

$$
\begin{pmatrix}
X_{11} & X_{12} & X_{13} & X_{14} & X_{15} \\
X_{21} & X_{22} & X_{23} & X_{24} & X_{25}
\end{pmatrix}
$$

Hence it is a $k$-algebra generated by the set

$$H = \{ [1 \, 2], [1 \, 3], [1 \, 4], [1 \, 5], [2 \, 3],
\quad [2 \, 4], [2 \, 5], [3 \, 4], [3 \, 5], [4 \, 5] \}.$$

We use the following notation for the product of two minors:

$$
\begin{pmatrix}
1 & 4 \\
2 & 3
\end{pmatrix}
:=
\begin{pmatrix}
1 & 4 \\
2 & 3
\end{pmatrix}
\cdot
\begin{pmatrix}
2 & 3
\end{pmatrix}
= (X_{11}X_{24} - X_{21}X_{14})(X_{12}X_{23} - X_{22}X_{13}).
$$

With this notation, the Plücker relation $p_{14}p_{23} - p_{13}p_{24} + p_{12}p_{34} = 0$ becomes:

$$
\begin{pmatrix}
1 & 4 \\
2 & 3
\end{pmatrix}
- \begin{pmatrix}
1 & 3 \\
2 & 4
\end{pmatrix}
+ \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}
= 0.
$$
Standard monomials in $G_{n,m}$

To identify a monomial in the letters of $H$, we use the tableaux

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$i_1$</td>
<td>$\ldots$</td>
<td>$i_n$</td>
</tr>
<tr>
<td>$j_1$</td>
<td>$\ldots$</td>
<td>$j_n$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$u_1$</td>
<td>$\ldots$</td>
<td>$u_n$</td>
</tr>
</tbody>
</table>

We call this a *standard* monomial if its rows are strictly increasing and its columns are weakly increasing.

Notice that if we multiply a standard monomial with a nonstandard one, we get a nonstandard monomial.

Claim: the standard monomials generate $G_{n,m}$ as a $k$-vector space.

To prove the claim, it is enough to show that a nonstandard product of two minors is a $k$-linear combination of standard monomials.
$G_{2,5}$: the straightening relations

Recall that the following Plücker relation

\[
\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} - \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 0
\]

holds in $G_{2,5}$.

The monomial $\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$ is nonstandard, while $\begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix}$ and $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ are both standard.

Hence we get the equation:

\[
\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix},
\]

which is called straightening relation for the nonstandard monomial $\begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix}$.
**$G_{2,5}$: the straightening relations**

Here is a list of all nonstandard products of two minors in $G_{2,5}$ together with their straightening relations:

\[
\begin{array}{ccc}
1 & 4 & \quad = \quad 1 & 3 & \quad - \quad 1 & 2 \\
2 & 3 & & 2 & 4 & & 3 & 4 \\
1 & 5 & \quad = \quad 1 & 3 & \quad - \quad 1 & 2 \\
2 & 3 & & 2 & 5 & & 3 & 5 \\
1 & 5 & \quad = \quad 1 & 4 & \quad - \quad 1 & 2 \\
2 & 4 & & 2 & 5 & & 4 & 5 \\
1 & 5 & \quad = \quad 1 & 4 & \quad - \quad 1 & 3 \\
3 & 4 & & 3 & 5 & & 4 & 5 \\
2 & 5 & \quad = \quad 2 & 4 & \quad - \quad 2 & 3 \\
3 & 4 & & 3 & 5 & & 4 & 5 \\
\end{array}
\]
$G_{2,5}$: ordering the minors

Let $i_1 \leq j_1, i_2 \leq j_2 \in H$. Set

\[
i_1 \leq j_1 \iff i_1 \leq j_1, i_2 \leq j_2.
\]

$\leq$ is a partial order on $H$.

It is not total: for example, $\begin{pmatrix} 1 & 4 \end{pmatrix}$ and $\begin{pmatrix} 2 & 3 \end{pmatrix}$ are not comparable.

Now let us look at one straightening relation:

\[
\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.
\]

Notice that $\begin{pmatrix} 1 & 4 \end{pmatrix}$ divides the l.h.s., while on the r.h.s. we have that:

- $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ is divisible by $\begin{pmatrix} 1 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 3 \end{pmatrix} < \begin{pmatrix} 1 & 4 \end{pmatrix}$,
- $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ is divisible by $\begin{pmatrix} 1 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \end{pmatrix} < \begin{pmatrix} 1 & 4 \end{pmatrix}$.

A similarly property holds for $\begin{pmatrix} 2 & 3 \end{pmatrix}$ and for the other straightening relations (I will refer to it as the “H2” condition).
\( G_{2,5} \): properties so far

\( G_{2,5} \), the homogeneous coordinate ring of Grass(2, \( k^5 \)):
- is a commutative \( k \)-algebra generated by

\[
H = \{12, 13, 14, 15, 23, 24, 25, 34, 35, 45\};
\]

- is generated as a \( k \)-vector space by the standard monomials;
- is endowed with straightening relations that enable us to express the nonstandard monomials as \( k \)-linear combinations of the standard ones;
- comes with a partial order on the set of generators \( H \) that satisfies the H2 condition.
Monomials

Let $H$ be a finite set.

**Definition**

A monomial on $H$ is an element of $\mathbb{N}^H$, i.e. a function $M : H \rightarrow \mathbb{N}$.

If we think of $H$ as the set of indeterminates in a polynomial ring $k[H]$, then we can associate to $M$ a monomial in the usual sense, namely

$$
\prod_{x \in H} x^{M(x)}.
$$

Given two monomials $M, N \in \mathbb{N}^H$, their product is defined by:

$$(MN)(x) := M(x) + N(x).$$

We say that $N$ divides $M$ if $N(x) \leq M(x)$ for every $x \in H$. 
Ideals of monomials

Definition

An *ideal of monomials* on $H$ is a subset $\Sigma \subseteq \mathbb{N}^H$ such that

$$M \in \Sigma, \ N \in \mathbb{N}^H \Rightarrow MN \in \Sigma.$$ 

Definition

A monomial $M$ is called *standard* with respect to the ideal $\Sigma$ if $M \notin \Sigma$.

Definition

A *generator* of an ideal $\Sigma$ is an element of $\Sigma$ which is not divisible by any other element of $\Sigma$.

The set of generators of an ideal $\Sigma$ is finite.
Hodge algebra

Consider

- $R$ a commutative ring;
- $A$ a commutative $R$-algebra;
- $H \subseteq A$ a finite partially ordered set;
- $\Sigma$ the ideal of monomials on $H$.

To each monomial $M$ on $H$, we can associate an element in $A$ that we still denote by $M$:

$$M := \prod_{x \in H} x^{M(x)}.$$
Hodge algebra

**Definition**

A is a *Hodge algebra* governed by \( \Sigma \) and generated by \( H \) if:

**H1**  
\( A \) is a free \( R \)-module on the standard monomials with respect to \( \Sigma \).

**H2**  
if \( N \in \Sigma \) is a generator and

\[
N = \sum_i r_i M_i, \quad 0 \neq r_i \in R,
\]

is the unique expression for \( N \in A \) as a linear combination of standard monomials (guaranteed by H1), then for each \( x \in H \)

\[
x|N \Rightarrow \forall i \exists y_i \in H \text{ such that } y_i|M_i \text{ and } y_i < x
\]

The relations in H2 are called the *straightening relations* of \( A \).
\[G_{2,5}: \text{the Hodge algebra structure}\]

\(G_{2,5}\) is a Hodge algebra over \(k\) generated by
\[
H = \{1\ 2, \ 1\ 3, \ 1\ 4, \ 1\ 5, \ 2\ 3, \ 2\ 4, \ 2\ 5, \ 3\ 4, \ 3\ 5, \ 4\ 5\}
\]
governed by the ideal of monomials
\[
\Sigma = \langle 1\ 4\ 2\ 3, 1\ 5\ 2\ 3, 1\ 5\ 2\ 4, 1\ 5\ 3\ 4, 2\ 5\ 3\ 4 \rangle
\]
with straightening relations given by
\[
\begin{align*}
1\ 4\ 2\ 3 &= 1\ 3\ 2\ 4 - 1\ 2\ 3\ 4, \\
1\ 5\ 2\ 3 &= 1\ 5\ 2\ 4 - 1\ 5\ 3\ 4, \\
1\ 5\ 3\ 4 &= 1\ 4\ 3\ 5 - 1\ 3\ 4\ 5
\end{align*}
\]
Hodge algebras arise as coordinate rings of algebraic varieties, for example

- Grassmannians;
- determinantal varieties;
- flag manifolds;
- Schubert varieties.
The discrete Hodge algebra

**Definition**

A Hodge algebra $A$ is called *discrete* if the right hand side of all straightening relations is 0, i.e. if $N = 0$ in $A$ for all $N \in \Sigma$.

If $R[H]$ is the polynomial ring over $R$ whose indeterminates are the elements of $H$, then

$$R[H]/\Sigma R[H]$$

is a discrete Hodge algebra and any other discrete Hodge algebra governed by $\Sigma$ is isomorphic to it.

To measure how far a Hodge algebra $A$ is from being discrete, we introduce the *indescrete part* $\text{Ind} A \subseteq H$ of $A$ defined as

$$\{x \in H \mid x \text{ appears in the r.h.s. of the straightening relations for } A\}$$
The simplification of Hodge algebras

Consider a multiplicative filtration of \( A \), i.e. a chain of ideals

\[
\mathcal{I} : A = I_0 \supseteq I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots
\]

such that \( I_p I_q \subseteq I_{p+q} \ \forall p, q \geq 0 \) and \( R \cap I_1 = \{0\} \).

To this filtration we can associate a new \( R \)-algebra by setting

\[
\text{gr}_\mathcal{I} A := A/I_1 \oplus I_1/I_2 \oplus I_2/I_3 \oplus \ldots
\]

**Theorem**

*If \( x \in H \) is a minimal element of \( \text{Ind} \ A \) and

\[
\mathcal{I} = \{x^n A\} : A \supseteq xA \supseteq x^2 A \supseteq x^3 A \supseteq \ldots,
\]

then \( \text{gr}_\mathcal{I} A \) is a Hodge algebra governed by \( \Sigma \) with

\[
\text{Ind}(\text{gr}_\mathcal{I} A) \subseteq \text{Ind} A \setminus \{x\}.
\]
The simplification of Hodge algebras

Corollary

If $A$ is a Hodge algebra governed by $\Sigma$, there is a sequence of elements $x_1, \ldots, x_n \in \text{Ind } A$ such that defining

$$A_n := A, \quad A_{i-1} := \text{gr}\{x^n_i A_i\} A_i \quad \forall i = 1, \ldots, n$$

we have:

- each $A_i$ is a Hodge algebra governed by $\Sigma$
- $x_i$ is minimal in $\text{Ind } A_i$
- $A_0$ is discrete.

This result may be viewed as a stepwise flat deformation, whose most general fiber is $A$ and whose most special fiber is the discrete Hodge algebra $R[H]/\Sigma R[H]$. 

Properties preserved under deformation

The previous result allows us to reduce many questions about a Hodge algebra $A$ to questions about more nearly discrete and therefore simpler Hodge algebras. In particular, many interesting properties that are satisfied by $A_0$ are preserved under the deformation and are also satisfied by $A$.

**Theorem**

- If $A_0$ is reduced, then $A$ is.
- If $A_0$ is Cohen-Macaulay, then $A$ is.
- If $A_0$ is Gorenstein, then $A$ is.
- If $R$ is a field or $\mathbb{Z}$, then $\dim A = \dim A_0$. 
Ideals generated by monomials

Recall that discrete Hodge algebras are isomorphic to $R[H]/\Sigma R[H]$, i.e. they are quotients of polynomial rings modulo ideals generated by monomials.

**Proposition**

Suppose $R$ is a domain. Let $\Sigma$ be an ideal of monomials and set $I = \Sigma R[H]$.

- $I$ is prime $\iff$ $\Sigma$ is generated by a subset of $H$.
- $I$ is radical $\iff$ $\Sigma$ is generated by square-free monomials.
- $I$ is primary $\iff$ whenever $x \in H$ divides a generator of $\Sigma$, there is a generator which is a power of $x$.
- The associated primes of $I$ are all generated by subsets of $H$. 
$G_{2,5}$: the discrete Hodge algebra

The discrete Hodge algebra $A_0$ obtained by deforming $G_{2,5}$, is the polynomial ring:

$$k \left[ \begin{array}{ccccccc} 1 & 2 & 1 & 3 & 1 & 4 & 1 & 5 & 2 & 3, \\
2 & 4 & 2 & 5 & 3 & 4 & 3 & 5 & 4 & 5 \end{array} \right]$$

modulo the ideal

$$I = \left( \begin{array}{ccccccc} 1 & 4 & 1 & 5 & 1 & 5 & 1 & 5 & 2 & 5, \\
2 & 3 & 2 & 3 & 2 & 4 & 3 & 4 & 3 & 4 \end{array} \right)$$

For example, we may notice that:

- $A_0$ is not a domain, since $I$ is not generated by a subset of $H$;
- $A_0$ is reduced, since $I$ is generated by square-free monomials.

As a consequence, we deduce that $G_{2,5}$ is reduced.
Simplicial complexes

Let $H$ be a finite set.

**Definition**

We say $\Delta$ is a *simplicial complex* with vertex set $H$, if $\Delta$ is a collection of subsets of $H$ (called *faces*) such that:

- $\forall x \in H$, $\{x\} \in \Delta$;
- $T \subseteq S \in \Delta \Rightarrow T \in \Delta$.

**Definition**

The *dimension* of a face $S$ in $\Delta$ is defined as $|S| - 1$.

**Definition**

The *dimension* of $\Delta$ is the maximum of the dimensions of its faces.
Ideals of monomials and simplicial complexes

Let $S \subseteq H$ and define a monomial $\chi_S \in \mathbb{N}^H$ by

$$\chi_S(x) := \begin{cases} 
1, & x \in S \\
0, & x \notin S
\end{cases}$$

Suppose $\Sigma \subseteq \mathbb{N}^H$ is an ideal of monomials such that

- $\Sigma$ is generated by square-free monomials;
- $\forall x \in H, \chi_x \notin \Sigma$.

If we define

$$\Delta := \{ S \subseteq H \mid \chi_S \notin \Sigma \},$$

then $\Delta$ is a simplicial complex with vertex set $H$.

**Proposition**

- The minimal primes of $A_0$ are generated by the complements of the maximal faces of $\Delta$.
- If $R$ is Noetherian, then $\dim A_0 = \dim R + \dim \Delta + 1$ and $\text{height } HA_0 = \dim \Delta + 1$. 
$G_{2,5}$: the dimension of $G_{2,5}$

Recall
\[ \Sigma = \langle \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 2 & 4 \end{pmatrix}, \begin{pmatrix} 1 & 5 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 2 & 5 \\ 3 & 4 \end{pmatrix} \rangle \]

If we construct $\Delta$ as before, the maximal faces are
- \{1 2, 1 3, 1 4, 1 5, 2 5, 3 5, 4 5\},
- \{1 2, 1 3, 1 4, 2 4, 2 5, 3 5, 4 5\},
- \{1 2, 1 3, 1 4, 2 4, 3 4, 3 5, 4 5\},
- \{1 2, 1 3, 2 3, 2 4, 2 5, 3 5, 4 5\},
- \{1 2, 1 3, 2 3, 2 4, 3 4, 3 5, 4 5\}.

Therefore
\[ \dim A_0 = \dim k + \dim \Delta + 1 = 0 + 6 + 1 = 7. \]

As a consequence, $\dim G_{2,5} = 7$. 
Wonderful posets

Let $H$ be a poset.

**Definition**

An element $y \in H$ is a *cover* of an element $x \in H$ if $x < y$ and no element of $H$ is properly between $x$ and $y$.

**Definition**

$H$ is *wonderful* if the following condition holds in the poset $H \cup \{-\infty, \infty\}$ obtained by adjoining least and greatest elements to $H$: if $y_1, y_2 < z$ are covers of an element $x$, then there is an element $y \leq z$ which is a cover of both $y_1$ and $y_2$. 
Some terminology for posets

Let $H$ be a finite poset.

**Definition**

A *chain* in $H$ is a totally ordered set $X \subseteq H$; its *length* is $|X| - 1$.

**Definition**

The *dimension* of $H$ is the maximum of the lengths of chains in $H$.

**Definition**

The *height* of an element $x \in H$, denoted $\text{ht } x$, is the maximum of the lengths of chains descending from $x$. 
\( G_{2,5} \): \( H \) is a wonderful poset

On the left is a diagram of the poset \( H \) for \( G_{2,5} \) (smaller elements appear on the bottom). We see that:

- \( H \) is a wonderful poset;
- \( \dim H = 6 \);
- the elements of height \( i \) are the ones appearing on the \( i \)-th row starting from the bottom and counting from 0.

Recall

\[
\Sigma = \left\langle \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 5 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 5 \\ 3 & 4 \end{bmatrix} \right\rangle
\]

Notice that \( \Sigma \) is generated by the products of the pairs of elements which are incomparable in the partial order on \( H \).
Wonderful posets and regular sequences

Let $A$ be a Hodge algebra generated by $H$ and governed by $\Sigma$.

**Definition**

$A$ is called *ordinal* if $\Sigma$ is generated by the products of the pairs of elements which are incomparable in the partial order on $H$.

**Theorem**

Let $A$ be an ordinal Hodge algebra and set

$$p_i = \sum_{x \in H : \text{ht } x = i} x.$$  

If $H$ is wonderful and $\dim H = n$, then $p_0, \ldots, p_n$ is a regular sequence.


$G_{2,5}$: a regular sequence

Let $A_0$ be the discrete Hodge algebra obtained from $G_{2,5}$.
The elements:

\begin{align*}
p_0 &= \begin{bmatrix} 1 & 2 \end{bmatrix} \\
p_1 &= \begin{bmatrix} 1 & 3 \end{bmatrix} \\
p_2 &= \begin{bmatrix} 1 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 3 \end{bmatrix} \\
p_3 &= \begin{bmatrix} 1 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 4 \end{bmatrix} \\
p_4 &= \begin{bmatrix} 2 & 5 \end{bmatrix} + \begin{bmatrix} 3 & 4 \end{bmatrix} \\
p_5 &= \begin{bmatrix} 3 & 5 \end{bmatrix} \\
p_6 &= \begin{bmatrix} 4 & 5 \end{bmatrix}
\end{align*}

form a regular sequence in $HA_0$.
Since $\dim A_0 = 7$, $A_0$ is Cohen-Macaulay.
As a consequence, $G_{2,5}$ is Cohen-Macaulay.