Generalized Tanisaki ideals and the cohomology of Hessenberg varieties

February 11, 2011

Tanisaki ideals

Let $n \in \mathbb{Z}_+$ and $\mu = (\mu_1, \ldots, \mu_n)$ a partition of n.

Let $\mathfrak{N}(\mu)$ be the set of nilpotent matrices in $Mat_n(\mathbb{C})$ whose Jordan canonical form has blocks of size $\mu_1, \ldots, \mu_n$.

De Concini and Procesi (1981) defined the ideal in $\mathbb{Q}[x_1, \ldots, x_n]$ of the schematic intersection $\overline{\mathfrak{N}(\mu)} \cap Diag_n(\mathbb{C})$.

Tanisaki (1982) simplified the construction of this ideal, now called (by some) Tanisaki ideal I_μ.

Garsia and Procesi (1992) described some properties of the quotient ring

$$R_\mu := \mathbb{Q}[x_1, \ldots, x_n]/I_\mu$$

and gave a combinatorial rule to construct a monomial basis of R_μ.

Springer varieties

Let $\mathfrak{F}$ be the full flag variety

$$\mathfrak{F} := \{V_1 \subseteq \ldots \subseteq V_n = \mathbb{C}^n \mid \forall i,\ \dim V_i = i\}.$$

Definition (Springer variety)

For $X \in \mathfrak{N}(\mu)$, define the *Springer variety*

$$\mathfrak{S}_X := \{\{V_1 \subseteq \ldots \subseteq V_n\} \in \mathfrak{F} \mid \forall i,\ XV_i \subseteq V_i\}.$$

If $Y = A^{-1}XA$ for some $A \in GL_n(\mathbb{C})$, then $\mathfrak{S}_Y \cong \mathfrak{S}_X$ (so we could replace the index X by the partition μ).

The cohomology ring $H^*(\mathfrak{S}_X)$ carries a symmetric group action. This action was discovered by Springer (1976) and later described by De Concini and Procesi.

In particular, $H^*(\mathfrak{S}_X) \cong R_\mu$.

Hessenberg varieties

Definition (Hessenberg function)

An n-tuple $h = (h_1, \ldots, h_n)$ is a *Hessenberg function* if

- $i \leqslant h_i \leqslant n$, $\forall i \in \{1, \ldots, n\}$,
- $h_i \leqslant h_{i+1}$, $\forall i \in \{1, \ldots, n-1\}$.

Example ($n = 6$)

$h = (2, 3, 3, 5, 5, 6)$.

Definition (Nilpotent Hessenberg variety)

Let $X \in Mat_n(\mathbb{C})$, X nilpotent.

$$\mathfrak{H}(X, h) := \{\{V_1 \subseteq \ldots \subseteq V_n\} \in \mathfrak{F} \mid \forall i \; XV_i \subseteq V_{h_i}\}.$$

Examples

- For $h = (n, \ldots, n)$, $\mathfrak{H}(X, h) = \mathfrak{F}$.
- For $h = (1, \ldots, n)$, $\mathfrak{H}(X, h) = \mathfrak{S}_X$.

Generalizing I_μ

Can we generalize I_μ to an ideal $I_{\mu,h}$ of $\mathbb{Q}[x_1, \ldots, x_n]$ such that

$$H^*(\mathfrak{H}(X,h)) \cong \mathbb{Q}[x_1, \ldots, x_n]/I_{\mu,h}?$$

Recent work by Mbirika and Tymoczko suggests this is possible at least for regular nilpotent Hessenberg varieties, corresponding to $\mu = (n)$.

The case $\mu = (n)$ corresponds to a matrix X with a single Jordan block. For most of this talk we fix $\mu = (n)$, so μ will be suppressed in the notation unless needed.

I present the construction and some properties of the generalized ideal I_h. After recalling what is known about $H^*(\mathfrak{H}(X,h))$, I will show how it is related to $\mathbb{Q}[x_1, \ldots, x_n]/I_h$.

Truncated elementary symmetric functions

Definition (Truncated elementary symmetric function)

For $S \subseteq \{1, \ldots, n\}$ and $d > 0$,

$$e_d(S) := \sum_{\{i_1 < \ldots < i_d\} \subseteq S} x_{i_1} \ldots x_{i_d}.$$

Example ($n = 4$)

$$e_2(1,2) = x_1 x_2$$
$$e_2(1,2,3) = x_1 x_2 + x_1 x_3 + x_2 x_3$$

A few conventions:

- if $d = 0$, then $e_d(S) = 1$ for all S;
- if $d < 0$ and $S \neq \varnothing$, then $e_d(S) = 0$;
- if $d > |S|$, then $e_d(S) = 0$.

h-Ferrer diagrams

Definition (h-Ferrer diagram)

Let $h = (h_1, \dots, h_n)$ be a Hessenberg function.

- Draw the diagram of the staircase partition $(1, \dots, n)$ flush right and bottom.
- Fill the bottom row with the numbers $h_1, \dots, h_n$ from left to right.
- Fill each columns with decreasing entries from bottom to top.

Example ($n = 6$)

$h = (2, 3, 3, 5, 5, 6)$

					1
				1	2
			2	2	3
		1	3	3	4
	2	2	4	4	5
2	3	3	5	5	6

Ideals I_h

For a Hessenberg function $h = (h_1, \ldots, h_n)$, introduce the set

$$\mathfrak{C}_h := \bigcup_{i=1}^{n} \{e_{h_i - r}(1, \ldots, h_i) \mid 0 \leqslant r \leqslant i-1\}.$$

The element $e_{h_i - r}(1, \ldots, h_i)$ in $\mathfrak{C}_h$ corresponds to the box in the i-th column and $(r+1)$-st row of the h-Ferrer diagram.

Example ($n = 4$)

$h = (3, 3, 3, 4)$

			1
		1	2
	2	2	3
3	3	3	4

$$\mathfrak{C}_h = \{e_3(1,2,3), e_2(1,2,3), e_1(1,2,3), e_4, e_3, e_2, e_1\}.$$

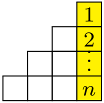
Ideals I_h

Definition (Generalized Tanisaki ideal)

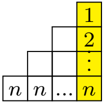
$$I_h := (\mathfrak{C}_h) \subseteq \mathbb{Q}[x_1, \ldots, x_n].$$

A few observations:

- for all h, $\mathfrak{C}_h \supseteq \{e_1, \ldots, e_n\}$;

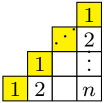


- if $h = (n, \ldots, n)$, then $\mathfrak{C}_h = (e_1, \ldots, e_n)$;



- if $h = (1, \ldots, n)$, then

$$\mathfrak{C}_h \supseteq (e_1(1), e_1(1,2), \ldots, e_1(1, \ldots, n))$$



so $I_h = (x_1, \ldots, x_n)$.

Poset on Hessenberg functions

Definition

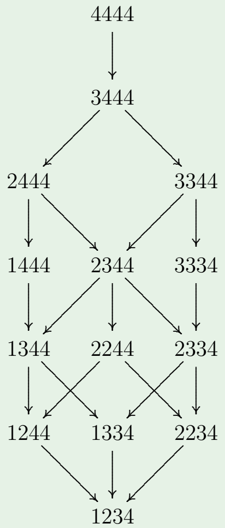
Let $h = (h_1, \ldots, h_n)$ and $h' = (h'_1, \ldots, h'_n)$ be Hessenberg functions.

$$h \geqslant h' \iff h_i \geqslant h'_i, \ \forall i.$$

We say $h > h'$ are *adjacent* if there is an edge connecting them in the Hasse diagram on Hessenberg functions, i.e.

- $h_{i_0} = h'_{i_0} + 1$, for some i_0;
- $h_{i_0} = h'_{i_0}$, for all $i \neq i_0$.

Example (n=4)



Poset on ideals I_h

Theorem

If $h > h'$, then $I_h \subset I_{h'}$.

Sketch of proof.

Enough to consider $h > h'$ adjacent, so $h_{i_0} = h'_{i_0} + 1$ for some i_0 and $h_i = h'_i$ for all $i \neq i_0$. Equivalently, the h- and h'-Ferrer diagrams are identical except in column i_0.

Need to show: the generators $e_{h_{i_0}-r}(1, \ldots, h_{i_0}) \in \mathfrak{C}_h$, for $0 \leqslant r \leqslant i_0 - 1$, also lie in $I_{h'}$.

$$e_{h_{i_0}-r}(1, \ldots, h_{i_0}) = e_{h'_{i_0}+1-r}(1, \ldots, h'_{i_0}+1) =$$
$$= x_{h'_{i_0}+1} \underbrace{e_{h'_{i_0}-r}(1, \ldots, h'_{i_0})}_{\in I_{h'}} + \underbrace{e_{h'_{i_0}+1-r}(1, \ldots, h'_{i_0})}_{\in I_{h'}}.$$

□

Poset on ideals I_h

It is not true in general that $\mathfrak{C}_h \subset \mathfrak{C}_{h'}$, when $h > h'$, even if they are adjacent.

Example ($n = 4$)

$h = (3, 4, 4, 4)$, $\mathfrak{C}_h = \{e_3(1, 2, 3) = x_1x_2x_3, e_1, e_2, e_3, e_4\}$
$h' = (2, 4, 4, 4)$, $\mathfrak{C}_{h'} = \{e_2(1, 2) = x_1x_2, e_1, e_2, e_3, e_4\}$

Lemma

Suppose $h > h'$ are adjacent with $h_{i_0} = h'_{i_0} + 1$. If $h_{i_0} = h'_k$ for some $k > i_0$, then $\mathfrak{C}_h \subset \mathfrak{C}_{h'}$.

Example ($n = 4$)

$h = (2, 3, 3, 4)$,

$$\mathfrak{C}_h = \{e_2(1, 2), e_1(1, 2, 3), e_2(1, 2, 3), e_3(1, 2, 3), e_1, e_2, e_3, e_4\}$$

$h' = (2, 2, 3, 4)$,

$$\mathfrak{C}_{h'} = \{e_1(1, 2), e_2(1, 2), e_1(1, 2, 3), e_2(1, 2, 3), e_3(1, 2, 3), e_1, e_2, e_3, e_4\}$$

Generator-containment sequences

Theorem

For each Hessenberg function $h > (1, \ldots, n)$, there exists at least one adjacent function h' with both $h > h'$ and $\mathfrak{C}_h \subset \mathfrak{C}_{h'}$.

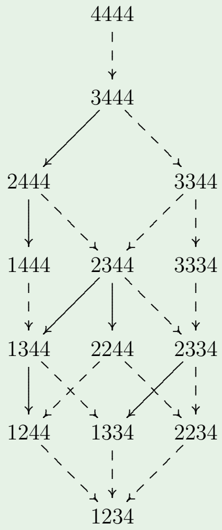
Definition (Generator-containment sequence)

Let $h = f_1 > \ldots > f_r = h'$ be a sequence of Hessenberg functions such that f_i and f_{i+1} are adjacent. If $\mathfrak{C}_{f_i} \subset \mathfrak{C}_{f_{i+1}}$, the sequence is a *generator-containment sequence* from h to h'.

Corollary

Given a Hessenberg function h, there exists a g.c.s. from h to $(1, \ldots, n)$. In particular, the g.c.s.'s form a spanning subgraph of the Hasse diagram on Hessenberg functions.

Example (n=4)



Reduced generating set for I_h

The generating set $\mathfrak{C}_h$ is often highly nonminimal.

Definition (Antidiagonal ideal)

$$I_h^{AD} := (\{e_{h_i-i+1}(1,\ldots,h_i) \mid 1 \leqslant i \leqslant n\}).$$

I_h^{AD} is 'generated by antidiagonal boxes' of the h-Ferrer diagram.

Theorem

$$I_h^{AD} = I_h.$$

Sketch of proof.

$I_h^{AD} \subseteq I_h$, so it's enough to show $\mathfrak{C}_h \subseteq I_h^{AD}$.

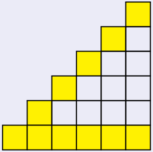
Claim: $e_{h_i}(1,\ldots,h_i) \in I_h^{AD}$ (bottom row generators)

- For $i = 1$, $e_{h_1}(1,\ldots,h_1) \in I_h^{AD}$ by definition.
- For $i > 1$, $e_{h_i}(1,\ldots,h_i) = x_1 \ldots x_{h_i} = x_{h_1+1} \ldots x_{h_i} e_{h_1}(1,\ldots,h_1)$.

Reduced generating set for I_h

Sketch of proof (continued).

<u>Claim</u>: $e_{h_i - r}(1, \ldots, h_i) \in I_h^{AD}$ (i-th column generators)



■ $\in I_h^{AD}$

- For $i = 2$, clear.
- For $i \geqslant 2$ and $h_{i+1} = h_i$, the $(i+1)$-st and i-th column coincide.
- For $i \geqslant 2$ and $h_{i+1} > h_i$, for all $s \in \{1, \ldots, i-1\}$

$$e_{h_{i+1}-s}(1, \ldots, h_{i+1}) = \sum_{t=0}^{h_{i+1}-h_i} e_t(h_i+1, \ldots, h_{i+1}) e_{h_{i+1}-s-t}(1, \ldots, h_i)$$

and $t \leqslant h_{i+1} - h_i$ implies $h_{i+1} - s - t \geqslant h_i - s$ so the right hand side lies in I_h^{AD}. □

Degree tuples

Definition (Degree tuple)

An n-tuple $\beta = (\beta_n, \ldots, \beta_1)$ is a *degree tuple* if

- $1 \leqslant \beta_i \leqslant i$, $\forall i \in \{1, \ldots, n\}$,
- $\beta_i - \beta_{i-1} \leqslant 1$, $\forall i \in \{2, \ldots, n\}$.

Example ($n = 6$)

$\beta = (1, 2, 1, 2, 2, 1)$.

Proposition

There is a bijective correspondence

$$\{\text{Hessenberg functions}\} \xleftrightarrow{F} \{\text{degree tuples}\}.$$

For $h = (h_1, \ldots, h_n)$, we have $\beta = F(h)$, where $\beta = (\beta_n, \ldots, \beta_1)$ and

$$\beta_i = i - |\{h_k \mid h_k < i\}|.$$

Graphical correspondence

Definition (Hessenberg diagrams)

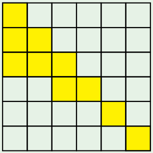
Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function.

- Draw an $n \times n$ square grid.
- Shade h_i boxes in column i, starting from the top.
- Remove $i - 1$ boxes in row i, starting from the right.

β_i is the number of shaded boxes in row i (starting from the bottom).

Example ($n = 6$)

$h = (3, 3, 4, 4, 5, 6)$



$\beta = (1, 1, 2, 3, 2, 1)$.

Truncated complete symmetric functions

Definition (Truncated complete symmetric function)

For $S \subseteq \{1, \ldots, n\}$ and $d > 0$,

$$\tilde{e}_d(S) := \sum_{\substack{\text{multisets} \\ \{i_1 \leqslant \ldots \leqslant i_d\} \subseteq S}} x_{i_1} \ldots x_{i_d}.$$

Example ($n = 4$)

$$\tilde{e}_3(3,4) = x_3^3 + x_3^2 x_4 + x_3 x_4^2 + x_4^3$$
$$\tilde{e}_2(2,3,4) = x_2^2 + x_2 x_3 + x_2 x_4 + x_3^2 + x_3 x_4 + x_4^2$$

A few conventions:

- if $d = 0$, then $\tilde{e}_d(S) = 1$ for all S;
- if $d < 0$ and $S \neq \varnothing$, then $\tilde{e}_d(S) = 0$.

Ideals J_h

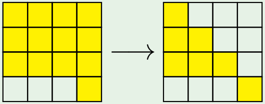
Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function and $\beta = (\beta_n, \ldots, \beta_1)$ the corresponding degree tuple.

Definition (Hessenberg basis ideal)

$$J_h := (\tilde{e}_{\beta_n}(n), \tilde{e}_{\beta_{n-1}}(n-1, n), \ldots, \tilde{e}_{\beta_1}(1, \ldots, n)) \subseteq \mathbb{Q}[x_1, \ldots, x_n].$$

Example ($n = 4$)

Let $h = (3, 3, 3, 4)$, then $\beta = (1, 3, 2, 1)$.



$$J_h = (\tilde{e}_1(4), \tilde{e}_3(3,4), \tilde{e}_2(2,3,4), \tilde{e}_1(1,2,3,4)) =$$
$$= (x_4, x_3^3 + x_3^2 x_4 + x_3 x_4^2 + x_4^3, x_2^2 + x_2 x_3 + x_2 x_4 + x_3^2 + x_3 x_4 + x_4^2,$$
$$x_1 + x_2 + x_3 + x_4).$$

Poset on degree tuples and ideals J_h

Definition

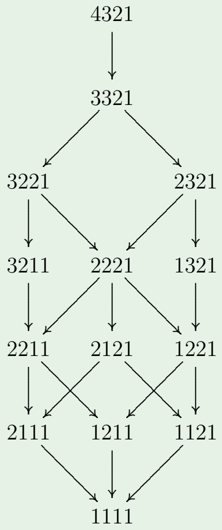
Let $\beta = (\beta_n, \ldots, \beta_1)$ and $\beta' = (\beta'_n, \ldots, \beta'_1)$ be degree tuples.

$$\beta \geqslant \beta' \iff \beta_i \geqslant \beta'_i, \ \forall i.$$

Facts

- The Hasse diagram for the partial order on degree tuples is the same as the one for Hesseberg functions.
- The bijection F between Hessenberg functions and degree tuples preserves the partial orders.
- If β, β' are degree tuples corresponding to the Hessenberg functions h, h' and $\beta > \beta'$, then $J_h \subset J_{h'}$.

Example (n=4)



Review of Groebner bases

For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, set $\mathbf{x}^\alpha := x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in R = \mathbb{Q}[x_1, \ldots, x_n]$.
Monomials in R are totally ordered lexicographically.

Definition

Suppose $f \in R$. Then $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$.

- (Leading monomial) $LM(f) = \max_{\alpha \in \mathbb{N}^n} \{\mathbf{x}^\alpha \mid c_\alpha \neq 0\}$.
- (Leading coefficient) $LC(f) = \max_{\alpha \in \mathbb{N}^n} \{c_\alpha \mid c_\alpha \neq 0\}$.
- (Leading term) $LT(f) = LC(f)LM(f)$.

Let $I \subseteq R$ be an ideal.

Definition (Leading term ideal)

$$(LT(I)) := (\{LT(f) \mid f \in I\}).$$

Definition (Groebner basis)

The set $\{g_1, \ldots, g_t\} \subset R$ is a *Groebner basis* of I if

$$(LT(I)) = (LT(g_1), \ldots, LT(g_t)).$$

Groebner basis of J_h

Proposition

Let $I = (g_1, \ldots, g_t) \subseteq R$. If, for every $i, j \in \{1, \ldots, t\}$ with $i \neq j$,

$$\operatorname{lcm}(LM(g_i), LM(g_j)) = LM(g_i)LM(g_j),$$

then $\{g_1, \ldots, g_t\}$ is a Groebner basis of I.

Corollary

The generators of J_h form a Groebner basis of J_h (with respect to the lexicographic monomial order).

Sketch of proof.

Let $\beta = (\beta_n, \ldots, \beta_1)$ be the degree tuple corresponding to h.
For $i \in \{1, \ldots, n\}$, set $f_i := \tilde{e}_{\beta_i}(i, \ldots, n)$, so that $J_h = (f_1, \ldots, f_n)$.
Observe that $LM(f_i) = x_i^{\beta_i}$. Then, for $i \neq j$,

$$\operatorname{lcm}(LM(f_i), LM(f_j)) = \operatorname{lcm}(x_i^{\beta_i}, x_j^{\beta_j}) = x_i^{\beta_i} x_j^{\beta_j} = LM(f_i)LM(f_j). \quad \square$$

A monomial basis of R/J_h

Proposition (Macaulay's Basis theorem)

Suppose $\{g_1, \ldots, g_t\}$ is a Groebner basis of I. Then

$$\{\mathbf{x}^\alpha \mid \forall i,\ LM(g_i) \nmid \mathbf{x}^\alpha\}$$

is a $\mathbb{Q}$-basis of R/I.

Theorem

Let h be a Hessenberg function and $\beta = (\beta_n, \ldots, \beta_1)$ the corresponding degree tuple. Then R/J_h has basis

$$\mathcal{B}_h := \{x_1^{\alpha_1} \ldots x_n^{\alpha_n} \mid 0 \leqslant \alpha_i \leqslant \beta_i - 1,\ \forall i \in \{1, \ldots, n\}\}.$$

In particular, $\dim_{\mathbb{Q}} R/J_h = \prod_{i=1}^n \beta_i$.

Sketch of proof.

The generators $f_1, \ldots, f_n$ of J_h form a Groebner basis.
$LM(f_i) = x_i^{\beta_i} \nmid x_1^{\alpha_1} \ldots x_n^{\alpha_n} \Rightarrow \alpha_i \leqslant \beta_i - 1$. □

Main theorem

Theorem (Mbirika, Tymoczko)

For any Hessenberg function h, $I_h = J_h$.

Sketch of proof.

- Prove $I_h = J_h$ when $h = (n, \ldots, n)$.
- Prove

$$\begin{aligned} e_d(1, \ldots, r) \in I_h &\iff \tilde{e}_d(r+1, \ldots, n) \in I_h, \\ \tilde{e}_d(r+1, \ldots, n) \in J_h &\iff e_d(1, \ldots, r) \in J_h. \end{aligned}$$

- Prove $I_h^{AD} \subseteq J_h$.
- Prove $J_h \subseteq I_h$. □

(h, μ)-fillings

Let μ be any partition of n.

Definition ((h, μ)-filling)

A filling of the Young diagram of μ with the numbers $1, \ldots, n$ is called an (h, μ)-*filling* if it satisfies the following rule:

$$\begin{array}{|c|c|}\hline k & j \\ \hline\end{array} \text{ is allowed only if } k \leqslant h_j.$$

Examples ($n = 3, \mu = (2, 1)$)

- For $h = (3, 3, 3)$, the (h, μ)-fillings are:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

- For $h = (2, 3, 3)$, the (h, μ)-fillings are:

$$\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 1 \\ \cline{1-1}\end{array} \quad \begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad\qquad \begin{array}{|c|c|}\hline 3 & 2 \\ \hline 1 \\ \cline{1-1}\end{array}$$

Dimension pairs

Let $h = (h_1, \ldots, h_n)$ be a Hessenberg function and μ a partition of n.

Definition (Dimension pair)

The pair (a, b) of entries of an (h, μ)-filling T is a *dimension pair* if:

1. $b > a$;
2. b is below a and in the same column or b is in any column strictly to the left of a;
3. if a box with filling c is adjacent and to the right of a, then $b \leqslant h_c$.

Examples $(n = 3, h = (2, 3, 3), \mu = (2, 1))$

$$\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1}\end{array}$$

(12) ✓ since $2 \leqslant h_3 = 3$
(13) ✗
(23) ✗

$$\begin{array}{|c|c|}\hline 2 & 1 \\ \hline 3 & \\ \cline{1-1}\end{array}$$

(12) ✓
(13) ✓
(23) ✗ since $3 > h_1 = 2$

The cohomology of $\mathfrak{H}(X, h)$

Theorem (Tymoczko)

Let $X \in \mathfrak{N}(\mu)$.

- If $i \in \mathbb{N}$ is odd, then $H^i(\mathfrak{H}(X, h)) = 0$.
- The dimension of $H^{2k}(\mathfrak{H}(X, h))$ is the number of (h, μ)-fillings T such that T has k dimension pairs.

Example $(n = 3, h = (2, 3, 3), \mu = (2, 1))$

(h, μ)-filling	$\begin{smallmatrix}1 & 2\\ 3 & \end{smallmatrix}$	$\begin{smallmatrix}1 & 3\\ 2 & \end{smallmatrix}$	$\begin{smallmatrix}2 & 3\\ 1 & \end{smallmatrix}$	$\begin{smallmatrix}2 & 1\\ 3 & \end{smallmatrix}$	$\begin{smallmatrix}3 & 2\\ 1 & \end{smallmatrix}$
dimension pairs	$(1, 3)$, $(2, 3)$	$(1, 2)$	$\varnothing$	$(1, 2)$, $(1, 3)$	$(2, 3)$

The Poincaré polynomial of $H^*(\mathfrak{H}(X, h))$ collects the information on the Betti numbers of $\mathfrak{H}(X, h)$:

$$P(t) = 1 + 2t^2 + 2t^4.$$

A map from (h,μ)-fillings to monomials

Fix a Hessenberg function $h = (h_1, \dots, h_n)$ and a partition μ of n.
If T is an (h,μ)-filling, denote by DP^T the set of dimension pairs of T.

Definition

$$\Phi \colon \{(h,\mu) - \text{fillings}\} \longrightarrow R = \mathbb{Q}[x_1, \dots, x_n]$$
$$T \longmapsto \prod_{(i,j) \in \mathrm{DP}^T} x_j$$

Denote by $\mathcal{A}_h(\mu)$ the image of Φ.

Example $(n = 3, h = (2,3,3), \mu = (2,1))$

(h,μ)-filling	$\begin{matrix}1&2\\3&\end{matrix}$	$\begin{matrix}1&3\\2&\end{matrix}$	$\begin{matrix}2&3\\1&\end{matrix}$	$\begin{matrix}2&1\\3&\end{matrix}$	$\begin{matrix}3&2\\1&\end{matrix}$
dimension pairs	$(1,3)$, $(2,3)$	$(1,2)$	$\varnothing$	$(1,2)$, $(1,3)$	$(2,3)$
monomial	x_3^2	x_2	1	x_2x_3	x_3

$\mathcal{A}_h(\mu) = \{1, x_2, x_3, x_2x_3, x_3^2\}$.

A map from (h, μ)-fillings to monomials

Let $M^{h,\mu}$ be the $\mathbb{Q}$-vector space with basis $\{(h, \mu) - \text{fillings}\}$.
Assume $\mu = (n)$. Recall R/J_h has basis

$$\mathcal{B}_h = \{x_1^{\alpha_1} \dots x_n^{\alpha_n} \mid 0 \leqslant \alpha_i \leqslant \beta_i - 1, \ \forall i \in \{1, \dots, n\}\},$$

where $\beta = (\beta_n, \dots, \beta_1)$ is the degree tuple corresponding to h.

Theorem (Mbirika)

For $\mu = (n)$, we have $\mathcal{A}_h(\mu) = \mathcal{B}_h$. In particular, the map Φ extends to an isomorphism of $\mathbb{Q}$-vector spaces:

$$M^{h,\mu} \xleftrightarrow{\cong} R/J_h.$$

This map is graded in the sense that it sends an (h, μ)-filling with k-dimension pairs to a monomial of degree k.

A similar result holds for $h = (1, \dots, n)$ and any μ, i.e. in the case of Springer varieties. In this case $\mathcal{A}_h(\mu)$ is the Garsia-Procesi basis of R_μ.

One last example

Example ($n = 4, h = (3,3,3,4), \beta = (1,3,2,1), \mu = (4)$)

By definition, $I_h = (e_3(1,2,3), e_2(1,2,3), e_1(1,2,3), e_4, e_3, e_2, e_1)$.
We have established that

$$I_h = I_h^{AD} = (e_3(1,2,3), e_2(1,2,3), e_1(1,2,3), e_1),$$
$$I_h = J_h = (\tilde{e}_1(4), \tilde{e}_3(3,4), \tilde{e}_2(2,3,4), \tilde{e}_1(1,2,3,4)).$$

The quotient $R/I_h = R/J_h$ has basis $\mathcal{B}_h = \{1, x_2, x_3, x_2x_3, x_3^2, x_2x_3^2\}$.

We can produce the same basis by listing all (h,μ)-fillings, finding their dimension pairs and constructing the corresponding monomials:

1 2 3 4	2 1 3 4	1 3 2 4	2 3 1 4	3 1 2 4	3 2 1 4
$\varnothing$	(12)	(23)	(12), (13)	(13), (23)	(12), (13), (23)
1	x_2	x_3	x_2x_3	x_3^2	$x_2x_3^2$

We can use R/I_h to recover the Betti numbers of $\mathfrak{H}(X,h)$.

Open questions

- Can we simultaneously generalize the Tanisaki ideal I_μ and the ideals I_h to a two-parameter family $I_{h,\mu}$, whose quotient $\mathbb{Q}[x_1, \ldots, x_n]/I_{h,\mu}$ is the cohomology ring of the Hessenberg variety for μ and h?
- Is there a ring isomorphism between $\mathbb{Q}[x_1, \ldots, x_n]/I_h$ and the cohomology of the regular nilpotent Hessenberg varieties (with rational coefficients)?
- In defining the antidiagonal ideal I_h^{AD}, we construct a reduced generating set for the ideal I_h. Is this reduced generating set minimal?
- The map $\Phi\colon \{(h, \mu) - \text{fillings}\} \to \mathcal{A}_h(\mu)$ admits an inverse when we fix $h = (1, \ldots, n)$ and let μ vary, and when we fix $\mu = (n)$ and let h vary. Is there an inverse map that incorporates both h and μ?

References

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