# Algorithms for irreducible decomposition of monomial ideals

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## 1 Irreducible Decomposition

A monomial ideal $I$ is an ideal of $k[x_1, \dots, x_n]$ generated by monomials.

**Example.** $I = (xy^3z, xy^2z^2, y^3z^2, y^2z^3)$

A monomial ideal is irreducible if it is generated by powers of some variables. An irreducible decomposition of a monomial ideal $I$ is an expression $I = I_1 \cap \dots \cap I_r$ with the $I_j$ irreducible.

**Example.** $I = (y^2) \cap (x, y^3, z^3) \cap (y^3, z^2) \cap (x, z^2) \cap (z)$

If $I_i \supseteq I_j$, then $I_i$ is redundant and can be dropped. When all the redundant components have been dropped, the irreducible decomposition is said to be irredundant.
You are probably familiar with the concept of primary decomposition of an ideal. Every irreducible monomial ideal is primary, so an irreducible decomposition is a primary decomposition. Depending on your applications, an irreducible decomposition might be better because:

- it is finer, since we are not lumping together irreducible ideals with the same radical;
- irredundant irreducible decompositions are unique (whereas primary decomposition need not be unique in general);
- algorithms for primary decomposition are resource and time consuming but specialized algorithms for irreducible decomposition of monomial ideals are available.

Some examples of applications of irreducible decomposition of monomial ideals are the Frobenius problem, the integer programming gap, the reverse engineering of biochemical networks, tropic convex hulls, tropical cyclic polytopes, secants of monomial ideals, differential powers of monomial ideals and joins of monomial ideal (for detailed references see [5]).

## 2 The Splitting Algorithm

This algorithm is described in [2].

**Lemma.** If $m$ is a minimal generator of a monomial ideal $I$ and $m = m_1m_2$, then

$$I = (I + (m_1)) \cap (I + (m_2)).$$

The splitting algorithm uses the lemma repeatedly to split a minimal generator into powers of different variables.

**Example.**

$$\begin{aligned}
I &= (xy^3z, xy^2z^2, y^3z^2, y^2z^3) = \\
&= (xy^3z, xy^2z^2, y^3z^2, y^2) \cap (xy^3z, xy^2z^2, y^3z^2, z^3) = \\
&= (y^2) \cap (xy^3z, xy^2z^2, y^3, z^3) \cap (xy^3z, xy^2z^2, z^2, z^3) = \\
&= (y^2) \cap (xy^2z^2, y^3, z^3) \cap (xy^3z, z^2) = \\
&= (y^2) \cap (xy^2, y^3, z^3) \cap (z^2, y^3, z^3) \cap (xy^3, z^2) \cap (z, z^2) = \\
&= (y^2) \cap (xy^2, y^3, z^3) \cap (y^3, z^2) \cap (xy^3, z^2) \cap (z) = \\
&= (y^2) \cap (x, y^3, z^3) \cap (y^2, z^3) \cap (y^3, z^2) \cap (x, z^2) \cap (y^3, z^2) \cap (z) = \\
&= (y^2) \cap (x, y^3, z^3) \cap (y^3, z^2) \cap (x, z^2) \cap (z)
\end{aligned}$$

After each split we reduce to a minimal set of generators. Notice how $(y^2, z^3) \supseteq (y^2)$ so it is redundant; also, $(y^3, z^2)$ appears twice so we can drop a copy.

The main drawback of this algorithm is that it does not output an irredundant decomposition. To get an irredundant decomposition we can eliminate the redundant components at the end or after each split. Either way, eliminating redundant components is time consuming and since the number of redundant components can be large, this algorithm is not vey efficient.

## 3 The Alexander Dual Algorithm

This algorithm is due to E. Miller and is described in [3]. Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ and $x^{\mathbf{b}} = x_1^{b_1} \ldots x_n^{b_n}$; we call $\mathbf{b}$ an exponent vector. Similarly, we write $\mathfrak{m}^{\mathbf{b}}$ for the irreducible monomial ideal $(x_1^{b_1}, \ldots, x_n^{b_n})$. If $\mathbf{a}$ and $\mathbf{b}$ are two exponent vectors and $\mathbf{a}$ is component wise bigger than $\mathbf{b}$, then we define the exponent vector $\mathbf{a} \setminus \mathbf{b}$ by setting

$$(\mathbf{a} \setminus \mathbf{b})_i := \begin{cases} 0, & b_i = 0 \\ a_i + 1 - b_i, & b_i \neq 0 \end{cases}$$

For an arbitrary monomial ideal $I$, take $\mathbf{a}_I$ to be the exponent vector on the least common multiple of the minimal generators of $I$; then define the Alexander dual of $I$ to be the ideal

$$I^{\vee} := (x^{\mathbf{a}_I \setminus \mathbf{b}} \mid \mathfrak{m}^{\mathbf{b}} \text{ is an irreducible component of } I).$$

Notice how the Alexander dual takes the irreducible components of $I$ to a set of minimal generators of $I^{\vee}$. This doesn't seem helpful since we don't know the irreducible components yet but it turns out that the Alexander dual also operates the other way around taking minimal generators to irreducible components.

**Theorem.**

$$I^{\vee} = \bigcap \{\mathfrak{m}^{\mathbf{a}_I \setminus \mathbf{b}} \mid x^{\mathbf{b}} \text{ is minimal generator of } I\}.$$

**Example.** $I = (xy^3z, xy^2z^2, y^3z^2, y^2z^3)$, $\mathbf{a}_I = (1, 3, 3)$

$$\begin{aligned} I^{\vee} &= (x, y, z^3) \cap (x, y^2, z^2) \cap (y, z^2) \cap (y^2, z) = \\ &= (x, xy^2, xz^2, xy, y^2, yz^2, xz^3, y^2z^3, z^3) \cap (y^2, yz, y^2z^2, z^2) = \\ &= (x, y^2, yz^2, z^3) \cap (y^2, yz, z^2) = \\ &= (xy^2, xyz, xz^2, y^2, y^2z, y^2z^2, y^2z^2, yz^2, yz^2, y^2z^3, yz^3, z^3) = \\ &= (xyz, xz^2, y^2, yz^2, z^3) \end{aligned}$$

Then $I = (x, y^3, z^3) \cap (x, z^2) \cap (y^2) \cap (y^3, z^2) \cap (z)$.

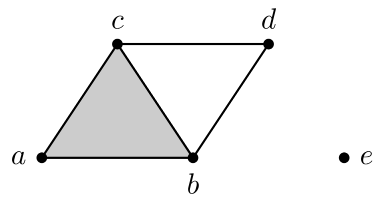
Why the algorithm works may be a little mysterious. The algorithm was inspired by an equivalent for square free monomial ideals.
A square free monomial ideal is a monomial ideal generated by square free monomials. Every square free monomial ideal in $k[x_1, \ldots, x_n]$ arises as the

Stanley-Reisner ideal of a unique simplicial complex $\Delta$ on $\{1,\ldots,n\}$; we denote this ideal $I_\Delta$. The generators of $I_\Delta$ are all the monomials whose support is not in $\Delta$. Equivalent formulations of the correspondence $\Delta \leftrightsquigarrow I_\Delta$ are given by the following bijections:

$$\begin{aligned}\{\text{minimal non faces of } \Delta\} &\leftrightsquigarrow \{\text{minimal generators of } I_\Delta\}\\ \{\text{faces of } \Delta\} &\leftrightsquigarrow \{\text{monomials in } k[x_1,\ldots,x_n]/I_\Delta\}\end{aligned}$$

**Example.**

$\Delta =$




$$I_\Delta = (ad, ae, be, ce, de, bcd)$$

The minimal generator $ad$ of $I_\Delta$ contains the variable $d$ which is not in the maximal face $\{a,b,c\}$ of $\Delta$; in other words, $ad$ contains a variable which is in the complement of $\{a,b,c\}$, i.e. in $\{d,e\}$. Therefore $ad \in (d,e)$ which is the irreducible ideal corresponding to the complement of the maximal face $\{a,b,c\}$ of $\Delta$. Notice the same holds with $ad$ and any maximal face of $\Delta$.

In general, it follows that a minimal generator of $I_\Delta$ belongs in the intersection of the irreducible ideals corresponding to the complement of the maximal faces of $\Delta$. The argument can be traced back and so we have the correspondence:

$$\{\text{complement of maximal faces of } \Delta\} \leftrightsquigarrow \{\text{irreducible components of } I_\Delta\}$$

and $I_\Delta^\vee = I_{\Delta^\vee}$, where $\Delta^\vee$ is the Alexander dual of the complex $\Delta$ defined by

$$\Delta^\vee := \big\{\{1,\ldots,n\} \setminus \sigma \mid \sigma \notin \Delta\big\}.$$

**Example.** For $I_\Delta = (ad, ae, be, ce, de, bcd)$, the maximal faces of $\Delta$ are $\{e\}$, $\{b,d\}$, $\{c,d\}$ and $\{a,b,c\}$. Hence

$$I = (a,b,c,d) \cap (a,c,e) \cap (a,b,e) \cap (d,e).$$

## 4 The Scarf Complex Algorithm

The Scarf complex was presented in [1] and is also discussed in [3]. Let $I = (m_1, \ldots, m_r)$ be a monomial ideal. For $U \subseteq \{1, \ldots, r\}$, set $m_U = \mathrm{lcm}(m_i, i \in U)$ (the monomial $m_U$ is called the *label* on $U$). The *Scarf complex* of $I$ is:

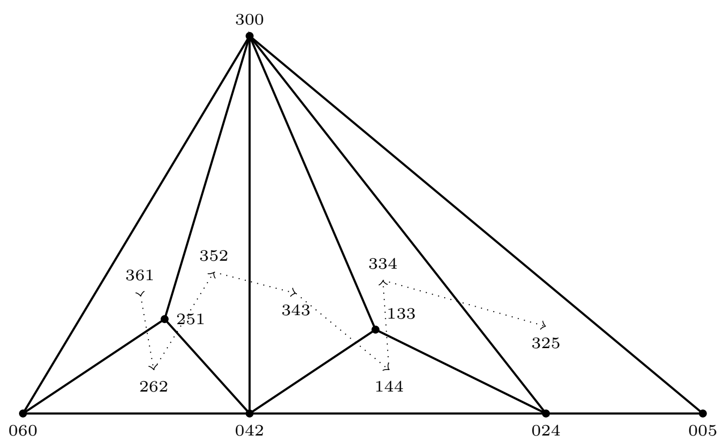
$$\Delta_I := \big\{U \subseteq \{1, \ldots, r\} \mid m_U = m_V \Rightarrow U = V\big\}$$

(i.e. $\Delta_I$ consists of sets of minimal generators of $I$ with unique labels).

**Lemma.** The Scarf complex $\Delta_I$ is a simplicial complex of dimension at most $n-1$.

**Example.** $J = (\underbrace{x^2y^5z}_{a}, \underbrace{xy^3z^3}_{b}, \underbrace{y^4z^2}_{c}, \underbrace{y^2z^4}_{d}) + (\underbrace{x^3}_{e}, \underbrace{y^6}_{f}, \underbrace{z^5}_{g})$

The vertices of the Scar complex $\Delta_J$ are labeled by the generators of $J$. In the picture, we write $ijk$ for the label $x^iy^jz^k$. We omit label for the edges since we are not going to need them later. For example, we have an edge labeled $y^4z^4$ between 042 and 024 because $\mathrm{lcm}(y^4z^2, y^2z^4) = y^4z^4$. Similarly we do not have an edge between 251 and 133 because $\mathrm{lcm}(x^2y^5z, xy^3z^3) = x^2y^5z^3 = \mathrm{lcm}(x^2y^5z, xy^3z^3, y^4z^2)$ and for the same reason there is no face with vertices 251, 133 and 042.




The Scarf complex $\Delta_J$

Here is the list of labels of the facets of $\Delta_J$:

$$\begin{aligned} x^3y^6z &= \mathrm{lcm}(a, e, f) \\ x^2y^6z^2 &= \mathrm{lcm}(a, c, f) \\ x^3y^5z^2 &= \mathrm{lcm}(a, c, e) \\ x^3y^4z^3 &= \mathrm{lcm}(b, c, e) \\ xy^4z^4 &= \mathrm{lcm}(b, c, d) \\ x^3y^3z^4 &= \mathrm{lcm}(b, d, e) \\ x^3y^2z^5 &= \mathrm{lcm}(d, e, g) \end{aligned}$$

A monomial ideal $I$ is called *(strongly) generic* if no two generators of $I$ raise the same variable to the same power. A monomial ideal is called *artinian* if it contains a power of each variable.

**Theorem.** The Scarf complex of a generic artinian monomial ideal is a triangulation of the $(n-1)$-simplex.

**Theorem.** Any generic artinian monomial ideal $I$ is the irredundant intersection of the ideals $\mathfrak{m}^{\mathbf{b}}$ where $x^{\mathbf{b}}$ is the label of a facet of the Scarf complex $\Delta_I$.

**Example.** $J = (x^2y^5z, xy^3z^3, y^4z^2, y^2z^4) + (x^3, y^6, z^5)$ is a generic artinian monomial ideal. The irredundant irreducible decomposition can be read off of the Scarf complex $\Delta_J$:

$$\begin{gathered} J = (x^3, y^6, z) \cap (x^2, y^6, z^2) \cap (x^3, y^5, z^2) \cap \\ \cap (x^3, y^4, z^3) \cap (x, y^4, z^4) \cap (x^3, y^3, z^4) \cap (x^3, y^2, z^5) \end{gathered}$$

**Remark.** The Scarf complex of a generic artinian monomial ideal can also be used to construct a minimal free resolution of the ideal. Constructing the entire Scarf complex requires a lot of least common multiple computations and therefore can be time and resource consuming. However, to read off the irreducible decomposition one only needs to know the facets of the complex. This information can be obtained efficiently by locating an initial facet and listing the others using a tree diagram and the algorithm developed by R. A. Milowski [4].

The Scarf complex method only works for generic Artinian monomial ideals. However any monomial ideal can be deformed to a generic monomial ideal by the following procedure. Fix an order on the variables: $x_1 > \ldots > x_n$. Then go through the variables in order and do the following:

1. arrange the generators of the ideal in decreasing order with respect to the degree of $x_i$;

2. whenever two monomials are tied in a non zero degree for $x_i$, break ties multiplying by $x_i$ one of the tied monomials and all the monomials that precede it in the previous arrangement.

**Example.** $I = (xy^3z, xy^2z^2, y^3z^2, y^2z^3)$
Start from $x$:

$$\overbrace{xy^3z}^{a} = \overbrace{xy^2z^2}^{b} > \overbrace{y^3z^2}^{c} = \overbrace{y^2z^3}^{d}$$
$$x^2y^3z > xy^2z^2 > y^3z^2 = y^2z^3$$

Then $y$:

$$\overbrace{x^2y^3z}^{a} = \overbrace{y^3z^2}^{c} > \overbrace{xy^2z^2}^{b} = \overbrace{y^2z^3}^{d}$$
$$x^2y^4z = y^4z^2 > xy^3z^2 > y^2z^3$$
$$x^2y^5z > y^4z^2 > xy^3z^2 > y^2z^3$$

Finally $z$:

$$\overbrace{y^2z^3}^{d} > \overbrace{xy^3z^2}^{b} = \overbrace{y^4z^2}^{c} > \overbrace{x^2y^5z}^{a}$$
$$y^2z^4 > xy^3z^3 > y^4z^2 > x^2y^5z$$

So we get the generic monomial ideal $(\underbrace{x^2y^5z}_{a}, \underbrace{xy^3z^3}_{b}, \underbrace{y^4z^2}_{c}, \underbrace{y^2z^4}_{d})$.

Next any generic monomial ideal can be turned into a generic Artinian monomial ideal by adding high enough powers of the variables.

**Example.** The ideal $(x^2y^5z, xy^3z^3, y^4z^2, y^2z^4)$ becomes generic Artinian after adding $(x^3, y^6, z^5)$.

Through the process above, any monomial ideal can be turned into a generic Artinian monomial ideal. The Scarf complex can then be employed to obtain an irreducible decomposition. This can in turn be used to recover a decomposition for the original ideal. First the powers of the variables that were added in can be discarded. Then we can use the following.

**Theorem.** Deformation "preserves" irreducible components.

However deforming back may introduce redundancy.

**Example.**
$I = (\underbrace{xy^3z}_{a}, \underbrace{xy^2z^2}_{b}, \underbrace{y^3z^2}_{c}, \underbrace{y^2z^3}_{d})$

$J = (\underbrace{x^2y^5z}_{a}, \underbrace{xy^3z^3}_{b}, \underbrace{y^4z^2}_{c}, \underbrace{y^2z^4}_{d}) + (\underbrace{x^3}_{e}, \underbrace{y^6}_{f}, \underbrace{z^5}_{g})$

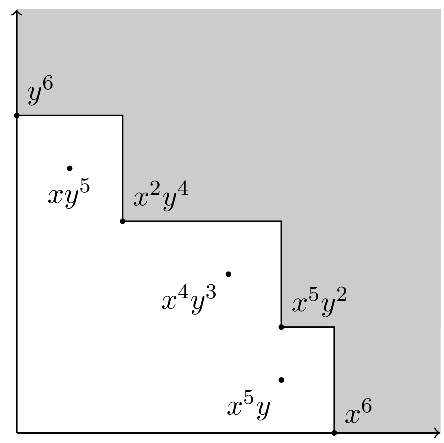
| | component of $J$ | component of $I$ |
|---|---|---|
| $x^3y^6z = \mathrm{lcm}(a, e, f)$ | $(x^3, y^6, z)$ | $(z)$ |
| $x^2y^6z^2 = \mathrm{lcm}(a, c, f)$ | $(x^2, y^6, z^2)$ | $(x, z^2)$ |
| $x^3y^5z^2 = \mathrm{lcm}(a, c, e)$ | $(x^3, y^5, z^2)$ | $(y^3, z^2)$ |
| $x^3y^4z^3 = \mathrm{lcm}(b, c, e)$ | $(x^3, y^4, z^3)$ | $(y^3, z^2)$ (same as previous) |
| $xy^4z^4 = \mathrm{lcm}(b, c, d)$ | $(x, y^4, z^4)$ | $(x, y^3, z^3)$ |
| $x^3y^3z^4 = \mathrm{lcm}(b, d, e)$ | $(x^3, y^3, z^4)$ | $(y^2, z^3)$ (contains next one) |
| $x^3y^2z^5 = \mathrm{lcm}(d, e, g)$ | $(x^3, y^2, z^5)$ | $(y^2)$ |

In conclusion, the Scarf complex method can be used to find a (possibly redundant) irreducible decomposition of a monomial ideal. Even so, the additional overhead introduced by deformation makes this algorithm less efficient if the ideal is not generic to begin with. If the ideal is generic, then this algorithm could be faster than the Alexander dual algorithm but it is still bound by the amount of memory available.

## 5 The Slice Algorithm

Finally we present the basic idea behind the slice algorithm by B. H. Roune [5]. Denote $R$ the polynomial ring $k[x_1, \ldots, x_n]$ and let $I$ be a monomial ideal in $R$. A monomial $m \in R$ is called a *maximal standard monomial* of $I$ if $m \notin I$ and $x_i m \in I$ for all $i = 1, \ldots, n$. The set of maximal standard monomials of $I$ is denoted $\mathrm{msm}(I)$.

**Example.** For $I = (x^6, x^5y^2, x^2y^4, y^6)$, $\mathrm{msm}(I) = \{x^5y, x^4y^3, xy^5\}$.
Each monomial corresponds to a point with integer coordinates. The shaded area corresponds to $I$. Multiplying by $x$ and $y$ pushes monomials to the right




and up. It is clear then that maximal standard monomials of $I$ correspond to those points outside of $I$ that are the closest to the inner corners of the shaded area.

**Remark.** The *socle* of $R/I$ is the $k$-vector subspace of $R/I$ defined by $\{\overline{m} \in R/I \mid x_i\overline{m} = 0\}$. The classes of the maximal standard monomials in $R/I$ form a basis for the socle of $R/I$.

The slice algorithm actually computes $\mathrm{msm}(I)$. This can be used to recover the irredundant irreducible decomposition of $I$ as we illustrate here. Choose an integer $t$ larger than the degree of any minimal generator of $I$ and define $\phi(x^m) = (x_i^{m_i+1} \mid m_i + 1 < t)$.

**Proposition.** The map $\phi$ is a bijection from $\mathrm{msm}(I + (x_1^t, \ldots, x_n^t))$ to the irreducible components of $I$.

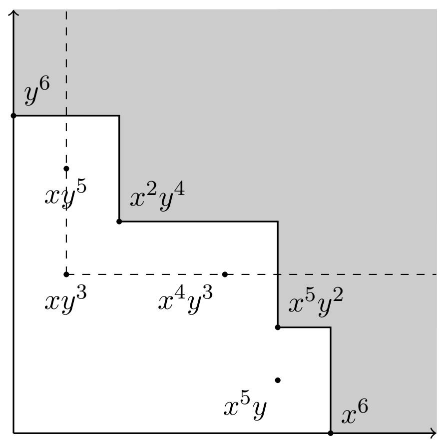
**Example.** $I = (x^6, x^5y^2, x^2y^4, y^6)$, $\mathrm{msm}(I) = \{x^5y, x^4y^3, xy^5\}$
Choose $t \geqslant 7$ then apply $\phi$ to get $I = (x^6, y^2) \cap (x^5, y^4) \cap (x^2, y^6)$.

The slice algorithm computes $\mathrm{msm}(I)$ by splitting it into two subsets called the *inner* and *outer slice*. Both slices depend on the choice of a single monomial $p$ called the *pivot*. The decomposition looks like this:

$$\mathrm{msm}(I) = (\mathrm{msm}(I) \cap (p)) \cup (\mathrm{msm}(I) \setminus (p)) \, .$$

**Example.** $I = (x^6, x^5y^2, x^2y^4, y^6)$, $p = xy^3$
It is clear from the picture that we have

$$\mathrm{msm}(I) \cap (p) = \{xy^5, x^4y^3\}$$
$$\mathrm{msm}(I) \setminus (p) = \{x^5y\}$$




The monomials from each slice are then regarded as maximal standard monomials from new ideals. Each slice is further split into smaller slices and the process repeats until the newly created slices are empty.

**Remark.** The basic version of the algorithm described here requires numerous steps to produce an answer even for a simple example like the one given above. The strength of the algorithm relies upon a number of optimizations like:

- different pivot selection strategies;
- monomial lower bounds on slice contents (to predict that an outer slice will be empty thus reducing to the computation of a single slice);
- simplification of slices;
- independence splits (if $I = I_1 + I_2$ is a monomial ideal in $k[\underline{x}, \underline{y}]$ with $I_1$ generated by monomials in the variables $\underline{x}$ and $I_2$ generated by monomials in the variables $\underline{y}$, then $\mathrm{msm}(I) = \mathrm{msm}(I_1) \cdot \mathrm{msm}(I_2)$);
- ad hoc strategies for dealing with monomials in two variables;
- and more.

Another advantage of this algorithm is that the inner and outer slices of a split can be computed in parallel to take advantage of systems with multiple processors.

## References

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