Free resolutions and representations with finitely many orbits

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Representations with finitely many orbits

- G complex linearly reductive group;
- V irreducible representation of G.

The pairs (G, V) such that the action $G \circlearrowright V$ has finitely many orbits were classified by V. Kac.

Example

- $\mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \circlearrowright \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$;
- orbits: $\mathcal{O}_r = \{\varphi : \mathbb{C}^n \to \mathbb{C}^m \mid \mathrm{rk}(\varphi) = r\}$
  i.e. matrices of given rank;
- orbit closures: $\overline{\mathcal{O}}_r = \{\varphi : \mathbb{C}^n \to \mathbb{C}^m \mid \mathrm{rk}(\varphi) \leqslant r\}$
  i.e. determinantal varieties.

Representation of a pair (X_n, α_k)

Theorem (Kac)

(X_n, α_k) Dynkin diagram with a distinguished node gives:

- $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, grading on the simple Lie algebra of type X_n;
- G_0, group of the Lie subalgebra $\mathfrak{g}_0$ (has diagram $X_n \backslash \alpha_k$).

The action $G_0 \circlearrowright \mathfrak{g}_1$ has finitely many orbits.

Example (A_{m+n-1}, α_m)

- $G_0 = \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C}) \times \mathrm{SL}_m(\mathbb{C})$
- $\mathfrak{g}_1 = \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^m)$

Representation of a pair (X_n, α_k)

Example (C_n, α_n)

- $G_0 = \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$
- $\mathfrak{g}_1 = \mathrm{Sym}_2(\mathbb{C}^n)$

Example (D_n, α_n)

- $G_0 = \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$
- $\mathfrak{g}_1 = \bigwedge^2(\mathbb{C}^n)$

Example ((E_n, α_2) for $n = 6, 7, 8$)

- $G_0 = \mathbb{C}^\times \times \mathrm{SL}_n(\mathbb{C})$
- $\mathfrak{g}_1 = \bigwedge^3(\mathbb{C}^n)$

Enumerating the orbits

Let $e \in \mathfrak{g}_1$ be nilpotent in $\mathfrak{g}$, and $C(e)$ be its conjugacy class in $\mathfrak{g}$. We have a decomposition into irreducible components:

$$C(e) \cap \mathfrak{g}_1 = C_1(e) \cup \ldots \cup C_{n(e)}(e)$$

Theorem (Vinberg)

The orbits of $G_0 \circlearrowright \mathfrak{g}_1$ are the irreducible components $C_i(e)$, for all choices of conjugacy classes $C(e)$ and all i, $1 \leqslant i \leqslant n(e)$.

Theorem (Vinberg)

The orbits of $G_0 \circlearrowright \mathfrak{g}_1$ correspond to some graded subalgebras of $\mathfrak{g}$.

The second result gives a recipe to enumerate all the orbits.

A wish list for the orbit closures

- $G_0 \circlearrowright \mathfrak{g}_1 = \mathcal{O}_0 \cup \ldots \cup \mathcal{O}_t$;
- $\mathfrak{g}_1 = \mathbb{A}^n_{\mathbb{C}}$ (complex affine space);
- $\overline{\mathcal{O}} \subseteq \mathbb{A}^n_{\mathbb{C}}$ affine algebraic variety.

Goal

Understand properties of the orbit closures $\overline{\mathcal{O}}$.

- Defining equations
- Containment
- Singular loci
- Cohen-Macaulay
- Gorenstein

Minimal free resolutions

- $A = \mathbb{C}[\mathbb{A}^n]$ is a polynomial ring,
- $\mathbb{C}[\overline{\mathcal{O}}] = A/I$, for some homogeneous ideal $I \subset A$.

We can achieve the goal by studying the minimal free resolution

$$\mathcal{F}_\bullet \colon F_0 \xleftarrow{d_1} F_1 \longleftarrow \ldots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n \longleftarrow 0$$

of $\mathbb{C}[\overline{\mathcal{O}}]$ as a graded A-module.

Moreover $\mathcal{F}_\bullet$ is G_0-equivariant, so

$$F_i = \bigoplus_{j \in \mathbb{Z}} U_j \otimes_{\mathbb{C}} A(-j),$$

for some representations U_j of G_0.

A bit of history

For the Lie algebras of classical type:

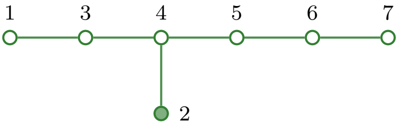
- Lascoux (1978), determinantal varieties (A_n);
- Józefiak, Pragacz, Weyman (1981), minors of symmetric and antisymmetric matrices;
- Lovett (2007), rank varieties (B_n, C_n, D_n).

For the Lie algebras of exceptional type:

- Kraśkiewicz, Weyman (2011), E_6, F_4 and G_2;
- Kraśkiewicz, Weyman (2013), E_7.

In some cases, Kraśkiewicz and Weyman only give the "expected resolution" of $\mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})]$, the coordinate ring of the normalization of the orbit closure.

(E_7, α_2): the representation



- $\mathfrak{g}(E_7) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$
- $\mathfrak{g}_0 = \mathbb{C} \oplus \mathfrak{sl}_7(\mathbb{C})$
- $G_0 = \mathbb{C}^\times \times \mathrm{SL}_7(\mathbb{C})$
- $\mathfrak{g}_1 = \bigwedge^3 \mathbb{C}^7$

(E_7, α_2): the orbits

The action $\mathbb{C}^\times \times \mathrm{SL}_7(\mathbb{C}) \circlearrowright \bigwedge^3 \mathbb{C}^7$ has 10 orbits:

- $\mathcal{O}_9$, the dense orbit i.e. $\overline{\mathcal{O}}_9 = \bigwedge^3 \mathbb{C}^7$;
- $\mathcal{O}_8$, with $\overline{\mathcal{O}}_8$ the hyperdiscrimant hypersurface;
- $\mathcal{O}_7$, with $\overline{\mathcal{O}}_7 = Sing(\overline{\mathcal{O}}_8) = \sigma_3(\overline{\mathcal{O}}_1)$;
- $\ldots$
- $\mathcal{O}_1$, the orbit of the highest weight vector with $\overline{\mathcal{O}}_1 = Cone(Gr(3,7))$;
- $\mathcal{O}_0$, the origin.

(E_7, α_2): the expected resolution for $\mathbb{C}[\overline{\mathcal{O}}_7]$

- $A = \mathrm{Sym}(\bigwedge^3 \mathbb{C}^7) = \mathbb{C}[x_{ijk} \mid 1 \leqslant i < j < k \leqslant 7]$.
- Expected resolution of $\mathbb{C}[\overline{\mathcal{O}}_7]$:

$$\mathbb{S}_{(0^7)}\mathbb{C}^7 \otimes A \leftarrow \mathbb{S}_{(3^4,2^3)}\mathbb{C}^7 \otimes A(-6) \leftarrow \mathbb{S}_{(4,3^5,2)}\mathbb{C}^7 \otimes A(-7) \leftarrow$$
$$\leftarrow \mathbb{S}_{(5^2,4^5)}\mathbb{C}^7 \otimes A(-10) \leftarrow \mathbb{S}_{(6,5^6)}\mathbb{C}^7 \otimes A(-12) \leftarrow 0$$

  where $\mathbb{S}_\lambda$ is the Schur functor associated to the partition λ.
- The Betti table:

	0	1	2	3	4
total:	1	35	48	21	7
0:	1	.	.	.	.
1:	.	.	.	.	.
2:	.	.	.	.	.
3:	.	.	.	.	.
4:	.	.	.	.	.
5:	.	35	48	.	.
6:	.	.	.	.	.
7:	.	.	.	21	.
8:	.	.	.	.	7

$(E_7, 2)$: the differential for $\overline{\mathcal{O}}_7$

$$d_2 : \mathbb{S}_{(4,3^5,2)}\mathbb{C}^7 \otimes A(-7) \longrightarrow \mathbb{S}_{(3^4,2^3)}\mathbb{C}^7 \otimes A(-6)$$

$$\left(\begin{matrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & x_{167} & 0 & x_{267} & \cdots \\
0 & 0 & x_{167} & 0 & x_{267} & 0 & 0 & 0 & 0 & \cdots \\
x_{167} & 0 & 0 & 0 & 0 & x_{367} & 0 & 0 & 0 & \cdots \\
0 & x_{267} & 0 & x_{367} & 0 & 0 & 0 & x_{467} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & -x_{157} & 0 & -x_{257} & \cdots \\
0 & 0 & -x_{157} & 0 & -x_{257} & 0 & 0 & 0 & 0 & \cdots \\
-x_{157} & 0 & 0 & 0 & 0 & -x_{357} & 0 & 0 & 0 & \cdots \\
0 & -x_{257} & 0 & -x_{357} & 0 & 0 & 0 & -x_{457} & 0 & \cdots \\
0 & 0 & x_{147} & 0 & x_{247} & 0 & x_{137} & 0 & x_{237} & \cdots \\
x_{147} & 0 & 0 & 0 & 0 & x_{347} & -x_{127} & 0 & 0 & \cdots \\
0 & x_{247} & 0 & x_{347} & 0 & 0 & 0 & 0 & -x_{127} & \cdots \\
-x_{137} & 0 & -x_{127} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & -x_{237} & 0 & 0 & -x_{127} & 0 & 0 & x_{347} & 0 & \cdots \\
0 & 0 & 0 & -x_{237} & 0 & -x_{137} & 0 & -x_{247} & 0 & \cdots \\
0 & 0 & 0 & 0 & 0 & 0 & x_{156} & 0 & x_{256} & \cdots \\
0 & 0 & x_{156} & 0 & x_{256} & 0 & 0 & 0 & 0 & \cdots \\
x_{156} & 0 & 0 & 0 & 0 & x_{356} & 0 & 0 & 0 & \cdots \\
0 & x_{256} & 0 & x_{356} & 0 & 0 & 0 & x_{456} & 0 & \cdots \\
0 & 0 & -x_{146} & 0 & -x_{246} & 0 & -x_{136} & 0 & -x_{236} & \cdots \\
-x_{146} & 0 & 0 & 0 & 0 & -x_{346} & x_{126} & 0 & 0 & \cdots \\
0 & -x_{246} & 0 & -x_{346} & 0 & 0 & 0 & 0 & x_{126} & \cdots \\
x_{136} & 0 & x_{126} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\
0 & x_{236} & 0 & 0 & x_{126} & 0 & 0 & -x_{346} & 0 & \cdots \\
0 & 0 & 0 & x_{236} & 0 & x_{136} & 0 & x_{246} & 0 & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{matrix}\right.$$

Constructing the complex

- Write an equivariant differential d_i explicitly in M2
- Compute syzygies of d_i and $d_i^\intercal$ with degree bounds
- Splice the resulting complexes

$$
\begin{array}{cccccccccc}
 & & & & F_{i-1} & \xleftarrow{d_i} & F_i & \longleftarrow & \mathcal{T}_\bullet & \longleftarrow 0 \\
0 & \longrightarrow & \mathcal{H}_\bullet & \longrightarrow & F_{i-1}^* & \xrightarrow{d_i^\intercal} & F_i^* & & & \\
\mathcal{F}_\bullet : & & \mathcal{H}_\bullet^* & \longleftarrow & F_{i-1} & \xleftarrow{d_i} & F_i & \longleftarrow & \mathcal{T}_\bullet & \longleftarrow 0
\end{array}
$$

Questions

- Does $\mathcal{F}_\bullet$ coincide with the expected resolution?
- Is $\mathcal{F}_\bullet$ exact?

Equivariant exactness criterion

$$\mathcal{F}_\bullet \colon F_0 \xleftarrow{d_1} F_1 \longleftarrow \dots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n \longleftarrow 0$$

Theorem (Buchsbaum-Eisenbud)

$\mathcal{F}_\bullet$ is exact $\iff \forall k = 1, \dots, n$

1. $\mathrm{rk}(F_k) = \mathrm{rk}(d_k) + \mathrm{rk}(d_{k+1})$;
2. $\mathrm{depth}(I(d_k)) \geqslant k$, where $I(d_k)$ is the ideal of A generated by the maximal non vanishing minors of d_k.

Proposition (G.)

$\mathcal{F}_\bullet$ is exact $\iff \forall k = 1, \dots, n$

1. $\mathrm{rk}(F_k) = \mathrm{rk}(d_k|_p) + \mathrm{rk}(d_{k+1}|_p)$ for p in the dense orbit;
2. $\mathrm{rk}(d_k)$ drops at orbit closures of codimension at least k.

(E_7, α_2): the resolution of $\mathbb{C}[\overline{\mathcal{O}_7}]$

$$A \leftarrow A(-6)^{35} \leftarrow A(-7)^{48} \leftarrow A(-10)^{21} \leftarrow A(-12)^{7} \leftarrow 0$$

orbit	$\mathrm{codim}(\overline{\mathcal{O}}_i)$	$\mathrm{rk}(d_1)$	$\mathrm{rk}(d_2)$	$\mathrm{rk}(d_3)$	$\mathrm{rk}(d_4)$
$\mathcal{O}_0$	35	0	0	0	0
$\mathcal{O}_1$	22	0	13	0	0
$\mathcal{O}_2$	15	0	20	0	1
$\mathcal{O}_3$	14	0	21	6	1
$\mathcal{O}_4$	10	0	25	3	3
$\mathcal{O}_5$	9	0	26	6	6
$\mathcal{O}_6$	7	0	28	6	4
$\mathcal{O}_7$	4	0	31	11	6
$\mathcal{O}_8$	1	1	34	14	7
$\mathcal{O}_9$	0	1	34	14	7

- $\mathrm{depth}(I(d_k)) = 4$ for $k = 1, 2, 3, 4$;
- $\overline{\mathcal{O}}_7$ is Cohen-Macaulay.

(E_7, α_2): containment and singular locus of $\overline{\mathcal{O}}_7$

The first differential d_1 contains equations for the orbit closure.

orbit	$\mathrm{rk}(d_1)$
$\mathcal{O}_0$	0
$\mathcal{O}_1$	0
$\mathcal{O}_2$	0
$\mathcal{O}_3$	0
$\mathcal{O}_4$	0
$\mathcal{O}_5$	0
$\mathcal{O}_6$	0
$\mathcal{O}_7$	0
$\mathcal{O}_8$	1
$\mathcal{O}_9$	1

Using representatives of each orbit, we can:

- determine orbit containment, by checking for vanishing of d_1:

$$\overline{\mathcal{O}}_7 = \mathcal{O}_0 \cup \ldots \cup \mathcal{O}_7;$$

- determine singular locus, via the Jacobian criterion:

$$Sing(\overline{\mathcal{O}}_7) = \mathcal{O}_0 \cup \ldots \cup \mathcal{O}_6.$$

The coordinate ring

We have a minimal free resolution $\mathcal{F}_\bullet \to R = A/I$, with $\mathcal{V}(I) = \overline{\mathcal{O}}$.

Question

Is R reduced? Equivalently, is I radical?

Proposition

A Noetherian ring R is reduced if and only if it satisfies the conditions (R_0) and (S_1).

Since $\overline{\mathcal{O}}$ is irreducible, I has a unique minimal prime $\mathfrak{p} = \sqrt{I}$.
Then:

- (S_1) means I has no embedded primes;
- (R_0) means $R_\mathfrak{p}$ is regular.

$$\mathcal{F}_\bullet \colon F_0 \xleftarrow{d_1} F_1 \longleftarrow \dots \longleftarrow F_{n-1} \xleftarrow{d_n} F_n \longleftarrow 0$$

$\mathcal{F}_\bullet$ is the minimal free resolution of $R = A/I$, with $\mathcal{V}(I) = \overline{\mathcal{O}}$.

Proposition (G.)

Assume $\operatorname{codim}(\overline{\mathcal{O}}) = c$.

- If $\operatorname{depth}(I(d_k)) > k$ for all $k > c$, then R satisfies (S_1).
- Let J be the Jacobian matrix of I and $x \in \mathcal{O}$.
  If $\operatorname{rk}(J|_x) = c$, then R satisfies (R_0).

Non normal orbits

The interactive method gives the minimal free resolution $\mathcal{F}_\bullet \to \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})]$.

$$\begin{array}{ccccccccc} & & & & \mathcal{F}_\bullet & \xrightarrow{\tilde{\pi}} & \mathcal{G}_\bullet & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{C}[\overline{\mathcal{O}}] & \hookrightarrow & \mathbb{C}[\mathcal{N}(\overline{\mathcal{O}})] & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array}$$

To present C:

- take $d_1 : F_1 \to F_0$;
- observe $F_0 = A \oplus F_0'$, with F_0' generated in degree $\geqslant 2$;
- $F_1 \to F_0'$ is a presentation of C.

To resolve $\mathbb{C}[\overline{\mathcal{O}}]$, take $\mathrm{cone}(\tilde{\pi})$.

State of the project

- E_6, F_4 and G_2
  - Results: http://arxiv.org/abs/1210.6410
  - M2 files for orbit closures, normalization and cokernels: http://www.mast.queensu.ca/~galetto/orbits
- E_7: Computationally intensive.
- E_8: ???