An algorithm for determining actions of semisimple Lie groups on free resolutions

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- $A = \mathbb{C}[x_1, \ldots, x_n]$
- M finitely generated graded A-module
- $F_\bullet$ minimal free resolution of M

Proposition

If G is a group which acts (reasonably) on A and M, then the action of G extends to $F_\bullet$.

Question

When $F_\bullet$ is determined computationally, can we also determine the action of G computationally?

Example

$A = \mathbb{C}[x,y,z]$, $M = A/(x,y,z)$

$$F_\bullet \colon A \xleftarrow{\begin{pmatrix} x & y & z \end{pmatrix}} A(-1)^3 \xleftarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} A(-2)^3 \xleftarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} A(-3) \leftarrow 0$$

If $V = \mathbb{C}^3 = \langle x,y,z \rangle$, then $A \cong \mathrm{Sym}(V)$ and $G = \mathrm{GL}(V)$ acts naturally on A and M.
Accounting for the G-action, $F_\bullet$ can be written as:

$$A \xleftarrow{d_1} V \otimes_{\mathbb{C}} A(-1) \xleftarrow{d_2} \bigwedge^2 V \otimes_{\mathbb{C}} A(-2) \xleftarrow{d_2} \bigwedge^3 V \otimes_{\mathbb{C}} A(-3) \leftarrow 0$$

1. The G-action may determine differentials.

Example

$d_2\colon \bigwedge^2 V \otimes_{\mathbb{C}} A(-2) \to V \otimes_{\mathbb{C}} A(-1)$ is determined by its degree 2 part, where the basis lives. Restrict to degree 2:

$$\bigwedge^2 V \longrightarrow V \otimes_{\mathbb{C}} A_1 \cong V \otimes_{\mathbb{C}} V \cong \bigwedge^2 V \oplus \mathrm{Sym}^2(V)$$

By Schur's lemma, there is only one such map up to scalars.

2. Determine the class

$$[M] = \sum_{i=0}^{n} (-1)^i [F_i]$$

of M in the Grothendieck group of the category of graded A-modules with a G-action.

A complex torus of rank m is a group $T \cong (\mathbb{C}^\times)^m$.

Theorem

If V is a finite dimensional representation of $T \cong (\mathbb{C}^\times)^m$, then

$$V \cong \bigoplus_{\alpha=(\alpha_1,\ldots,\alpha_m)\in\mathbb{Z}^m} V_\alpha,$$

where

$$V_\alpha = \{v \in V \mid \forall t = (t_1, \ldots, t_n) \in T, t \cdot v = t_1^{\alpha_1} \ldots t_m^{\alpha_m} v\}.$$

A non zero $v \in V_\alpha$ is called a *weight vector* with weight α.

Theorem

If G is a complex reductive group, it contains a maximal torus T and every finite dimensional representation of G is uniquely determined by the weights of T.

Example

$V = \mathbb{C}^3 = \langle e_1, e_2, e_3 \rangle$

$$\mathrm{GL}(V) \cong \mathrm{GL}_3(\mathbb{C}) \supset T = \left\{ \begin{pmatrix} t_1 & 0 & 0 \\ 0 & t_2 & 0 \\ 0 & 0 & t_3 \end{pmatrix} \middle| t_1, t_2, t_3 \in \mathbb{C}^\times \right\} \cong (\mathbb{C}^\times)^3$$

$$\forall t \in T \qquad t \cdot e_1 = t_1 e_1 = t_1^1 t_2^0 t_3^0 e_1 \qquad \mathrm{wt}(e_1) = (1,0,0)$$

$$t \cdot e_2 = t_2 e_2 = t_1^0 t_2^1 t_3^0 e_1 \qquad \mathrm{wt}(e_2) = (0,1,0)$$

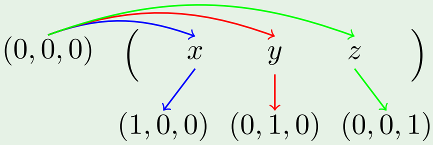
$$t \cdot e_3 = t_3 e_3 = t_1^0 t_2^0 t_3^1 e_1 \qquad \mathrm{wt}(e_3) = (0,0,1)$$

- V has weights $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$. Any representation with these 3 weights is isomorphic to V.
- $\bigwedge^2 V$ has weights $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$.

- Assume the variables of A are weight vectors.
- Assume $0 \leftarrow M \leftarrow F_0 \xleftarrow{d_1} F_1$ is a minimal presentation and the matrix of d_1 is written in a basis of weight vectors of F_0.
- Calculate a weight from each column in the matrix of d_1.

Example

- $A = \mathrm{Sym}(V)$, $V = \mathbb{C}^3 = \langle x, y, z \rangle$ and $G = \mathrm{GL}(V)$
- $\mathrm{wt}(x) = (1,0,0)$, $\mathrm{wt}(y) = (0,1,0)$, $\mathrm{wt}(z) = (0,0,1)$
- $d_1 = \begin{pmatrix} x & y & z \end{pmatrix}$ and G acts trivially on F_0



So $F_1 \cong V \otimes_{\mathbb{C}} A(-1)$.

- Consider any term ordering on A.
- For $\mathcal{F} = \{f_1, \ldots, f_s\}$ a basis of F, use *term over position up*:

$$t_1 f_i > t_2 f_j \iff t_1 > t_2 \text{ or } t_1 = t_2 \text{ and } i > j.$$

- Calculate a weight from the leading term of each column.

Example

Let $x > y > z$.

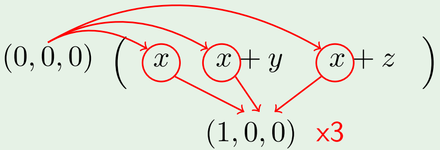
$$\begin{matrix} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{matrix} \begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}$$

$$(1,1,0) \quad (1,0,1) \quad (0,1,1)$$

So $F_2 \cong \bigwedge^2 V \otimes_{\mathbb{C}} A(-2)$.

If the matrix is not written in bases of weight vectors, we may run into trouble!

Example



There is no representation of $\mathrm{GL}_3(\mathbb{C})$ with these weights.

We remedy this by changing basis in the domain so that all columns have different leading terms.

Theorem (G.)

Let T be a torus and $\varphi\colon E \to F$ be a minimal T-equivariant homogeneous map of free A-modules. Suppose:

- $\Phi = (\Phi_1 | \dots | \Phi_r)$ *is the matrix of φ w.r.t bases $\mathcal{E} = \{e_1, \dots, e_r\}$ of E and $\mathcal{F} = \{f_1, \dots, f_s\}$ of F;*
- *F is equipped with term over position up w.r.t $\mathcal{F}$;*
- $\mathrm{LT}(\Phi_1) < \dots < \mathrm{LT}(\Phi_r)$*;*
- *F admits a basis of weight vectors $\tilde{\mathcal{F}} = \{\tilde{f}_1, \dots, \tilde{f}_s\}$ s.t. the change of basis from $\mathcal{F}$ to $\tilde{\mathcal{F}}$ is upper triangular.*

Then:

- *E admits a basis of weight vectors $\tilde{\mathcal{E}} = \{\tilde{e}_1, \dots, \tilde{e}_r\}$ s.t. the change of basis from $\mathcal{E}$ to $\tilde{\mathcal{E}}$ is upper triangular;*
- *if* $\mathrm{LT}(\Phi_i) = t f_j$, $\mathrm{wt}(\tilde{e}_i) = \mathrm{wt}(t) + \mathrm{wt}(\tilde{f}_j)$.