Equivariant resolutions of De Concini-Procesi ideals

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**Theorem**

Let $R = \mathbb{C}[x_1, \ldots, x_n]$, $M$ be a finitely generated graded $R$-module and

$$F_\bullet: F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \xleftarrow{\ldots} F_{n-1} \xleftarrow{\partial_n} F_n \xleftarrow{0}$$

a graded minimal free resolution of $M$. Let $G$ be a linearly reductive group acting on $R$ and $M$

- $\mathbb{C}$-linearly,
- preserving degrees ($\deg(g \cdot m) = \deg(m)$),
- preserving products ($g \cdot (rm) = (g \cdot r)(g \cdot m)$).

Then $G$ acts on each $F_i$ and the action commutes with $\partial_i$.

The resolution $F_\bullet$ is said to be $G$-equivariant.
Equivariant resolutions

**Proposition**

*If $F_\bullet$ is an equivariant resolution with the action of a group $G$, each $F_i$ can be written as $V_i \otimes_\mathbb{C} R$, where $V_i$ is a finite dimensional graded representation of $G$."

To understand an equivariant resolution, we need to identify the isomorphism class of each $V_i$ as a representation. This information can be used to:

- give a representation theoretic/combinatorial interpretation of Betti numbers;
- refine invariants (such as the Hilbert series);
- describe the differentials in a resolution.
De Concini-Procesi ideals

- $\mu = (\mu_1, \ldots, \mu_t)$ partition of $n$
- $X_\mu$ $n \times n$ unipotent matrix whose Jordan canonical form has blocks of size $\mu_1, \ldots, \mu_t$

**Definition**

The Springer fiber associated to $\mu$ is

$$F_\mu := \{ V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n \mid \dim_{\mathbb{C}}(V_i) = i, X_\mu(V_i) \subseteq V_i \}. $$

**Theorem (De Concini-Procesi, 1981)**

$$H^*(F_\mu) \cong R/I_{\tilde{\mu}}, \text{ where } \tilde{\mu} \text{ is the conjugate partition of } \mu. $$

$I_{\tilde{\mu}}$ is the De Concini-Procesi ideal associated with $\tilde{\mu}$. 
\begin{itemize}
    \item $\mu = (\mu_1, \ldots, \mu_t)$ partition of $n$
    \item $\mathcal{D}$ set of $n \times n$ diagonal matrices;
    \item $\mathcal{N}_\mu$ set of $n \times n$ nilpotent matrices whose Jordan canonical form has blocks of size $\mu_1, \ldots, \mu_t$
\end{itemize}

**Theorem (Kraft, 1981)**

\[ \mathcal{D} \times_{\mathbb{A}^n \mathbb{C}} \mathcal{N}_\mu \cong \text{Spec}(R/I_\mu) \]

$I_\mu$ is the De Concini-Procesi ideal associated with $\mu$. 
Facts & questions about DCP ideals

Facts

- The action of the symmetric group $\mathfrak{S}_n$ permuting the variables of $R = \mathbb{C}[x_1, \ldots, x_n]$ stabilizes $I_\mu$.
- The representation theoretic structure of $R/I_\mu$ is well understood (Garsia-Procesi, 1992).
- De Concini-Procesi ideals have recently been used in computation of Hilbert series for certain artinian Gorenstein ideals (Geramita-Hoefel-Wehlau, 2014).

Questions

- Can we describe a minimal generating set of $I_\mu$?
- What are the graded Betti numbers of $I_\mu$?
- Can we describe an $\mathfrak{S}_n$-equivariant resolution of $I_\mu$?
The case of hook partitions

Proposition (Biagioli-Faridi-Rosas 2007)

For $1 \leq d \leq n$ and $\mu = (n - d + 1, 1^{d-1})$,

- $I_\mu = (e_1, \ldots, e_{d-1}) + (x_{i_1} \ldots x_{i_d}) = E_{n,d} + I_{n,d}$, where the $e_i$ are elementary symmetric polynomials;
- $\{e_1, \ldots, e_{d-1}\} \cup \{x_{i_1} \ldots x_{i_d}\}$ is a minimal generating set of $I_\mu$;
- a minimal free resolution of $I_{n,d}$ gives one of $I_\mu$ via iterated mapping cones.

Facts

- $I_{n,d}$ is $\mathfrak{S}_n$-stable.
- Each $e_i$ is $\mathfrak{S}_n$-invariant.
- An equivariant resolution of $I_{n,d}$ gives one of $I_\mu$ via iterated mapping cones.
Sample resolutions

$I_{n,d}$ has a linear $\mathfrak{S}_n$-equivariant minimal free resolution:

$$0 \leftarrow I_{n,d} \leftarrow F_0 \leftarrow F_1 \leftarrow \ldots \leftarrow F_{n-d} \leftarrow 0$$

so $F_i \cong U^{n,d}_i \otimes_{\mathbb{C}} R(-d-i)$.

Examples

- For $d = n$,
  $$0 \leftarrow I_{n,n} \leftarrow [n] \leftarrow 0$$

- For $d = 1$, $F_\bullet$ is a Koszul complex, so
  $$U^{n,1}_i \cong \bigwedge^{i+1} \mathbb{C}^n \cong [n-i-1, 1^{i+1}] \oplus [n-i, 1^i].$$

- For $d = n-1$,
  $$0 \leftarrow I_{n,n-1} \leftarrow [n] \oplus [n-1, 1] \leftarrow [n-1, 1] \leftarrow 0$$
Main result

**Theorem (G.)**

$I_{n,d}$ has an $\mathfrak{S}_n$-equivariant minimal free resolution of the form

\[ U_0^{n,d} \otimes R(-d) \leftarrow \ldots \leftarrow U_i^{n,d} \otimes R(-d - i) \leftarrow \ldots \]

where, for all $0 \leq i \leq n - d$,

\[ U_i^{n,d} \cong \text{Ind}_{\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}}^{\mathfrak{S}_n} ([d, 1^i] \otimes [n - d - i]) \]

**Facts**

- $U_i^{n,d}$ is multiplicity free.
- $\text{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} (U_i^{n,d}) \cong U_i^{n-1,d} \oplus U_i^{n-1,d-1} \oplus U_{i-1}^{n-1,d}$. 
**Proposition (G.)**

\[ U_{i}^{n,d} \text{ has a basis consisting of standard Young tableaux on } (d, 1^i) \text{ with entries from } \{1, \ldots, n\}. \]

**Example (n=5,d=2,i=2)**

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\( U_{2}^{4,1} \)

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\( U_{1}^{4,2} \)

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