Equivariant resolutions of De Concini-Procesi ideals

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Equivariant resolutions

Theorem

Let $R = \mathbb{C}[x_1, \dots, x_n]$, M *be a finitely generated graded* R*-module and*

$$F_\bullet \colon F_0 \xleftarrow{\partial_1} F_1 \xleftarrow{\partial_2} F_2 \leftarrow \dots \leftarrow F_{n-1} \xleftarrow{\partial_n} F_n \leftarrow 0$$

a graded minimal free resolution of M.
Let G *be a linearly reductive group acting on* R *and* M

- $\mathbb{C}$*-linearly,*
- *preserving degrees* ($\deg(g \cdot m) = \deg(m)$),
- *preserving products* ($g \cdot (rm) = (g \cdot r)(g \cdot m)$).

Then G *acts on each* F_i *and the action commutes with* ∂_i.

The resolution $F_\bullet$ is said to be G-*equivariant*.

Equivariant resolutions

Proposition

If $F_\bullet$ is an equivariant resolution with the action of a group G, each F_i can be written as $V_i \otimes_{\mathbb{C}} R$, where V_i is a finite dimensional graded representation of G.

To understand an equivariant resolution, we need to identify the isomorphism class of each V_i as a representation.
This information can be used to:

- give a representation theoretic/combinatorial interpretation of Betti numbers;
- refine invariants (such as the Hilbert series);
- describe the differentials in a resolution.

De Concini-Procesi ideals

- $\mu = (\mu_1, \ldots, \mu_t)$ partition of n
- X_μ $n \times n$ unipotent matrix whose Jordan canonical form has blocks of size $\mu_1, \ldots, \mu_t$

Definition

The Springer fiber associated to μ is

$$\mathcal{F}_\mu := \{V_0 \subset V_1 \subset \ldots \subset V_n = \mathbb{C}^n \mid \dim_{\mathbb{C}}(V_i) = i, X_\mu(V_i) \subseteq V_i\}.$$

Theorem (De Concini-Procesi, 1981)

$H^*(\mathcal{F}_\mu) \cong R/I_{\tilde{\mu}}$, where $\tilde{\mu}$ is the conjugate partition of μ.

$I_{\tilde{\mu}}$ is the De Concini-Procesi ideal associated with $\tilde{\mu}$.

De Concini-Procesi ideals

- $\mu = (\mu_1, \ldots, \mu_t)$ partition of n
- $\mathcal{D}$ set of $n \times n$ diagonal matrices;
- $\mathcal{N}_\mu$ set of $n \times n$ nilpotent matrices whose Jordan canonical form has blocks of size $\mu_1, \ldots, \mu_t$

Theorem (Kraft, 1981)

$$\mathcal{D} \times_{\mathbb{A}^{n^2}_{\mathbb{C}}} \overline{\mathcal{N}}_\mu \cong \operatorname{Spec}(R/I_\mu)$$

I_μ is the De Concini-Procesi ideal associated with μ.

Facts & questions about DCP ideals

Facts

- The action of the symmetric group $\mathfrak{S}_n$ permuting the variables of $R = \mathbb{C}[x_1, \ldots, x_n]$ stabilizes I_μ.
- The representation theoretic structure of R/I_μ is well understood (Garsia-Procesi, 1992).
- De Concini-Procesi ideals have recently been used in computation of Hilbert series for certain artinian Gorenstein ideals (Geramita-Hoefel-Wehlau, 2014).

Questions

- Can we describe a minimal generating set of I_μ?
- What are the graded Betti numbers of I_μ?
- Can we describe an $\mathfrak{S}_n$-equivariant resolution of I_μ?

The case of hook partitions

Proposition (Biagioli-Faridi-Rosas 2007)

For $1 \leqslant d \leqslant n$ *and* $\mu = (n-d+1, 1^{d-1})$*,*

- $I_\mu = (e_1, \ldots, e_{d-1}) + (x_{i_1} \ldots x_{i_d}) = E_{n,d} + I_{n,d}$*, where the* e_i *are elementary symmetric polynomials;*
- $\{e_1, \ldots, e_{d-1}\} \cup \{x_{i_1} \ldots x_{i_d}\}$ *is a minimal generating set of* I_μ*;*
- *a minimal free resolution of* $I_{n,d}$ *gives one of* I_μ *via iterated mapping cones.*

Facts

- $I_{n,d}$ is $\mathfrak{S}_n$-stable.
- Each e_i is $\mathfrak{S}_n$-invariant.
- An equivariant resolution of $I_{n,d}$ gives one of I_μ via iterated mapping cones.

Sample resolutions

$I_{n,d}$ has a linear $\mathfrak{S}_n$-equivariant minimal free resolution:

$$0 \longleftarrow I_{n,d} \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \ldots \longleftarrow F_{n-d} \longleftarrow 0$$

so $F_i \cong U_i^{n,d} \otimes_{\mathbb{C}} R(-d-i)$.

Examples

- For $d = n$,

$$0 \longleftarrow I_{n,n} \longleftarrow [n] \longleftarrow 0$$

- For $d = 1$, $F_\bullet$ is a Koszul complex, so

$$U_i^{n,1} \cong \bigwedge^{i+1} \mathbb{C}^n \cong [n-i-1, 1^{i+1}] \oplus [n-i, 1^i].$$

- For $d = n-1$,

$$0 \longleftarrow I_{n,n-1} \longleftarrow [n] \oplus [n-1, 1] \longleftarrow [n-1, 1] \longleftarrow 0$$

Main result

Theorem (G.)

$I_{n,d}$ has an $\mathfrak{S}_n$-equivariant minimal free resolution of the form

$$U_0^{n,d} \otimes R(-d) \leftarrow \ldots \leftarrow U_i^{n,d} \otimes R(-d-i) \leftarrow \ldots$$

where, for all $0 \leqslant i \leqslant n-d$,

$$U_i^{n,d} \cong \operatorname{Ind}_{\mathfrak{S}_{d+i} \times \mathfrak{S}_{n-d-i}}^{\mathfrak{S}_n} \left([d, 1^i] \otimes [n-d-i] \right).$$

Facts

- $U_i^{n,d}$ is multiplicity free.
- $\operatorname{Res}_{\mathfrak{S}_{n-1}}^{\mathfrak{S}_n} (U_i^{n,d}) \cong U_i^{n-1,d} \oplus U_i^{n-1,d-1} \oplus U_{i-1}^{n-1,d}$.

Combinatorial interpretation of Betti numbers

Proposition (G.)

$U_i^{n,d}$ has a basis consisting of standard Young tableaux on $(d, 1^i)$ with entries from $\{1, \ldots, n\}$.

Example (n=5,d=2,i=2)