Symmetric Complete Intersections

Federico Galetto
(joint with A.V. Geramita and D. Wehlau)

McMaster University

AMS Spring Southeastern Sectional Meeting
University of Georgia, Athens
March 6, 2016
Let $R = \mathbb{C}[x_1, \ldots, x_n]$ with the standard grading.

**Definition**

A homogeneous ideal $I \subseteq R$ generated by a regular sequence $f_1, \ldots, f_c$ is called a *complete intersection ideal*.

**Facts**

Let $I = (f_1, \ldots, f_c) \subseteq R$ be a CI ideal. Let $d_i = \deg(f_i)$.

- $R/I$ is resolved by a Koszul complex.
- The Betti numbers of $R/I$ depend only on $d_1, \ldots, d_c$.
- The Hilbert series of $R/I$ depends only on $d_1, \ldots, d_c$. 
The symmetric group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables. This action

- is linear,
- preserves degrees,
- and is compatible with multiplication.

If $I \subseteq R$ is an $\mathfrak{S}_n$-stable ideal, then $\mathfrak{S}_n$ acts on $R/I$ and on its minimal free resolutions.

**Question**

What is the classification of $\mathfrak{S}_n$-stable CI ideals in terms of commutative algebra AND representation theory?
Examples of $\mathfrak{S}_n$-stable CI ideals

Example

$R = \mathbb{C}[x_1, x_2, x_3, x_4]$.

- $I = (e_1, e_2, e_3, e_4)$, where $e_i$ denotes the $i$-th elementary symmetric polynomial
- $I = (e_1, e_2, e_3, v)$, where

\[
v = \prod_{1 \leq i < j \leq 4} (x_i - x_j)
\]

is the Vandermonde determinant

- $I = (x_1^d, x_2^d, x_3^d, x_4^d)$, for $d \geq 1$
- $I = ((x_1 - x_3)(x_2 - x_4), (x_1 - x_2)(x_3 - x_4))$
Representations of the symmetric group

Let us review the basics of the representation theory of $\mathfrak{S}_n$ over $\mathbb{C}$.

**Facts**

- Every finite dimensional representation of $\mathfrak{S}_n$ is a direct sum of irreducible representations (with multiplicity).
- The irreducible representations $S^\lambda$ of $\mathfrak{S}_n$ are in bijection with partitions $\lambda$ of $n$.
- The standard Young tableaux of shape $\lambda$ form a basis of $S^\lambda$.
- If $T$ is a standard Young tableau and $i, j$ are entries in the same column of $T$, then the transposition $(i \ j)$ acts on $T$ by $(i \ j)T = -T$. 
Lemma

Let \( \varphi: S^\lambda \rightarrow R_d \) be a non-zero map of \( \mathfrak{S}_n \)-representations. If \( T \) is a standard tableau of shape \( \lambda \) containing 1 and 2 in the same column, then \( \varphi(T) \) is a polynomial divisible by \( x_1 - x_2 \).

Since two polynomials with a common factor do not form a regular sequence, the \( S^\lambda \) that generate a regular sequence must have \( \lambda \) equal to one of the following:

\[
\begin{array}{c|c|c|c|c}
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\end{array}
\quad
\begin{array}{c|c|c|c|c}
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\end{array}
\quad
\begin{array}{c|c|c|c|c}
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\end{array}
\quad
\begin{array}{c|c|c|c|c}
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\end{array}
\quad
\begin{array}{c|c|c|c|c}
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\ & \ & \ & \ & \\
\hline
\end{array}
\]
Theorem (Galetto-Geramita-Wehlau)

Suppose $I \subseteq R$ is an $\mathfrak{S}_n$-stable complete intersection ideal. Then $I/\mathfrak{m}I$ is isomorphic to one of the following:

1. a direct sum of up to $n$ trivial representations $S^{(n)}$;
2. a direct sum of an alternating representations $S^{(1^m)}$ and up to $n - 1$ trivial representations;
3. a direct sum of a standard representation $S^{(n-1,1)}$ and up to one trivial representation;
4. for $n = 4$, a direct sum of the irreducible representation $S^{(2,2)}$ and up to two trivial representations.
Examples of $\mathfrak{S}_n$-stable CI ideals

Example

- $I = (e_1, e_2, e_3, e_4)$ is generated by symmetric polynomials.
- $I = (e_1, e_2, e_3, v)$ is generated by one alternating polynomial together with symmetric polynomials.
- $I = (x_1^d, x_2^d, x_3^d, x_4^d)$, for $d \geq 1$. We have:

$$\langle x_1^d, x_2^d, x_3^d, x_4^d \rangle \cong \langle p_d \rangle \oplus S^{(3,1)},$$

where $p_d = x_1^d + x_2^d + x_3^d + x_4^d$ is symmetric.

- $I = ((x_1 - x_3)(x_2 - x_4), (x_1 - x_2)(x_3 - x_4))$ has generators that span a copy of $S^{(2,2)}$. 
Graded characters

If $V$ is a representation of $\mathfrak{S}_n$, then the character of $V$ is the function $\chi_V : \mathfrak{S}_n \to \mathbb{C}$ defined by $\chi_V(\sigma) = \text{trace}(\sigma : V \to V)$.

**Definition**

If $I \subset R$ is an $\mathfrak{S}_n$-stable ideal, the graded character of $R/I$ is

$$\chi_{R/I}(\sigma, t) = \sum_{d \in \mathbb{Z}} \chi_{(R/I)_d}(\sigma) t^d.$$ 

**Question**

If $I \subset R$ is an $\mathfrak{S}_n$-stable CI ideal, then what is the graded character of $R/I$?

Note that $\chi_{R/I}(1_{\mathfrak{S}_n}, t) = H_{R/I}(t)$ (\(=\) Hilbert series of $R/I$).
Character formulas

We provide character formulas for each case of our classification.

Example

$I = (x_1^2, x_2^2, x_3^2, x_4^2) \subseteq R = \mathbb{C}[x_1, x_2, x_3, x_4]$

\[
\chi_{R/I} = \chi^{(4)} + (\chi^{(4)} + \chi^{(3,1)})t + (\chi^{(4)} + \chi^{(3,1)} + \chi^{(2,2)})t^2 \\
+ (\chi^{(4)} + \chi^{(3,1)})t^3 + \chi^{(4)}t^4,
\]

where $\chi^\lambda$ is the character of $S^\lambda$. Compare with

\[
H_{R/I} = 1 + 4t + 6t^2 + 4t^3 + t^4.
\]

Note also that the socle of $R/I$ is a trivial representation.