Symmetric Complete Intersections

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Complete Intersections

Let $R = \mathbb{C}[x_1, \ldots, x_n]$ with the standard grading.

Definition

A homogeneous ideal $I \subseteq R$ generated by a regular sequence $f_1, \ldots, f_c$ is called a *complete intersection ideal*.

Facts

Let $I = (f_1, \ldots, f_c) \subseteq R$ be a CI ideal. Let $d_i = \deg(f_i)$.

- R/I is resolved by a Koszul complex.
- The Betti numbers of R/I depend only on $d_1, \ldots, d_c$.
- The Hilbert series of R/I depends only on $d_1, \ldots, d_c$.

Symmetric Complete Intersections

The symmetric group $\mathfrak{S}_n$ acts on $R = \mathbb{C}[x_1, \ldots, x_n]$ by permuting the variables. This action

- is linear,
- preserves degrees,
- and is compatible with multiplication.

If $I \subseteq R$ is an $\mathfrak{S}_n$-stable ideal, then $\mathfrak{S}_n$ acts on R/I and on its minimal free resolutions.

Question

What is the classification of $\mathfrak{S}_n$-stable CI ideals in terms of commutative algebra AND representation theory?

Examples of $\mathfrak{S}_n$-stable CI ideals

Example

$R = \mathbb{C}[x_1, x_2, x_3, x_4]$.

- $I = (e_1, e_2, e_3, e_4)$, where e_i denotes the i-th elementary symmetric polynomial
- $I = (e_1, e_2, e_3, v)$, where

$$v = \prod_{1 \leqslant i < j \leqslant 4} (x_i - x_j)$$

  is the Vandermonde determinant
- $I = (x_1^d, x_2^d, x_3^d, x_4^d)$, for $d \geqslant 1$
- $I = ((x_1 - x_3)(x_2 - x_4), (x_1 - x_2)(x_3 - x_4))$

Representations of the symmetric group

Let us review the basics of the representation theory of $\mathfrak{S}_n$ over $\mathbb{C}$.

Facts

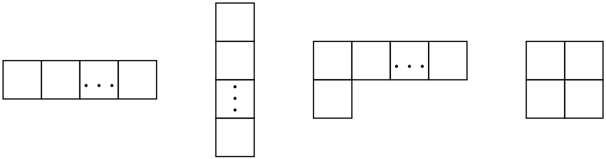
- Every finite dimensional representation of $\mathfrak{S}_n$ is a direct sum of irreducible representations (with multiplicity).
- The irreducible representations S^λ of $\mathfrak{S}_n$ are in bijection with partitions λ of n.
- The standard Young tableaux of shape λ form a basis of S^λ.
- If T is a standard Young tableau and i, j are entries in the same column of T, then the transposition $(i\ j)$ acts on T by
$$(i\ j)T = -T.$$

Representations not generating CIs

Lemma

Let $\varphi\colon S^{\lambda} \to R_d$ be a non-zero map of $\mathfrak{S}_n$-representations. If T is a standard tableau of shape λ containing 1 and 2 in the same column, then $\varphi(T)$ is a polynomial divisible by $x_1 - x_2$.

Since two polynomials with a common factor do not form a regular sequence, the S^{λ} that generate a regular sequence must have λ equal to one of the following:



Classification of symmetric CIs

Theorem (Galetto-Geramita-Wehlau)

Suppose $I \subseteq R$ is an $\mathfrak{S}_n$-stable complete intersection ideal. Then $I/\mathfrak{m}I$ is isomorphic to one of the following:

1. *a direct sum of up to n trivial representations $S^{(n)}$;*
2. *a direct sum of an alternating representations $S^{(1^n)}$ and up to $n-1$ trivial representations;*
3. *a direct sum of a standard representation $S^{(n-1,1)}$ and up to one trivial representation;*
4. *for $n = 4$, a direct sum of the irreducible representation $S^{(2,2)}$ and up to two trivial representations.*

Examples of $\mathfrak{S}_n$-stable CI ideals

Example

- $I = (e_1, e_2, e_3, e_4)$ is generated by symmetric polynomials.
- $I = (e_1, e_2, e_3, v)$ is generated by one alternating polynomial together with symmetric polynomials.
- $I = (x_1^d, x_2^d, x_3^d, x_4^d)$, for $d \geqslant 1$. We have:

$$\langle x_1^d, x_2^d, x_3^d, x_4^d \rangle \cong \langle p_d \rangle \oplus S^{(3,1)},$$

  where $p_d = x_1^d + x_2^d + x_3^d + x_4^d$ is symmetric.
- $I = ((x_1 - x_3)(x_2 - x_4), (x_1 - x_2)(x_3 - x_4))$ has generators that span a copy of $S^{(2,2)}$.

Graded characters

If V is a representation of $\mathfrak{S}_n$, then the character of V is the function $\chi_V \colon \mathfrak{S}_n \to \mathbb{C}$ defined by $\chi_V(\sigma) = \operatorname{trace}(\sigma \colon V \to V)$.

Definition

If $I \subset R$ is an $\mathfrak{S}_n$-stable ideal, the *graded character* of R/I is

$$\chi_{R/I}(\sigma, t) = \sum_{d \in \mathbb{Z}} \chi_{(R/I)_d}(\sigma) t^d.$$

Question

If $I \subset R$ is an $\mathfrak{S}_n$-stable CI ideal, then what is the graded character of R/I?

Note that $\chi_{R/I}(1_{\mathfrak{S}_n}, t) = H_{R/I}(t)$ (= Hilbert series of R/I).

Character formulas

We provide character formulas for each case of our classification.

Example

$I = (x_1^2, x_2^2, x_3^2, x_4^2) \subseteq R = \mathbb{C}[x_1, x_2, x_3, x_4]$

$$\begin{aligned}\chi_{R/I} = \chi^{(4)} + (\chi^{(4)} + \chi^{(3,1)})t + (\chi^{(4)} + \chi^{(3,1)} + \chi^{(2,2)})t^2 \\ + (\chi^{(4)} + \chi^{(3,1)})t^3 + \chi^{(4)}t^4,\end{aligned}$$

where χ^λ is the character of S^λ. Compare with

$$H_{R/I} = 1 + 4t + 6t^2 + 4t^3 + t^4.$$

Note also that the socle of R/I is a trivial representation.