Distinguishing k-configurations

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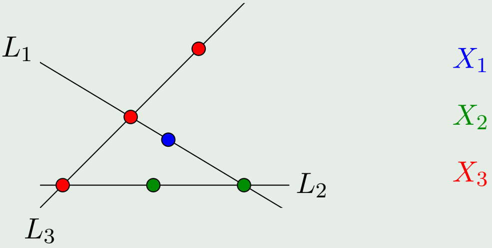
Definition (k-configuration)

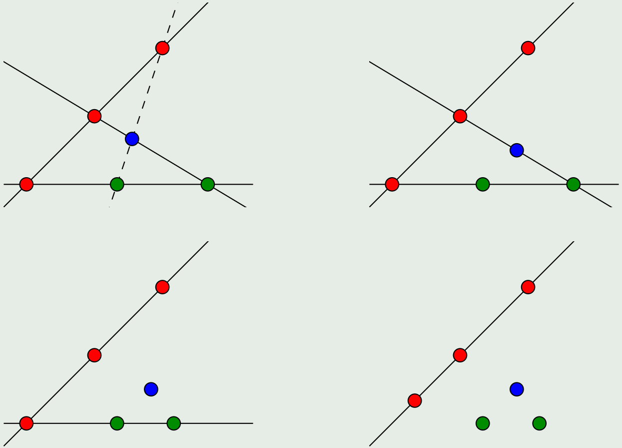
A set of points X is a *k-configuration* of type $(d_1, \ldots, d_s)$ if

1. $X = \bigsqcup_{i=1}^{s} X_i$, with $|X_i| = d_i$ and $d_1 < d_2 < \cdots < d_s < \infty$,
2. $X_i \subset L_i$ for some line L_i,
3. $X_i \cap L_j = \varnothing$ for all $i < j$.

Example (A k-configuration of type $(1, 2, 3)$)



Example (Different k-configurations of type $(1, 2, 3)$)



The number of lines through 3 points is different.

We introduce an equivalence relation on vectors in $\mathbb{C}^3$:

$$(a_0, a_1, a_2) \sim (b_0, b_1, b_2) \iff \exists \lambda \in \mathbb{C}^\times : (a_0, a_1, a_2) = \lambda(b_0, b_1, b_2)$$

Definition (Projective plane)

$$\mathbb{P}^2 := (\mathbb{C}^3 \setminus \{(0,0,0)\}) / \sim$$

A point in $\mathbb{P}^2$ is an equivalence class $[a_0 : a_1 : a_2]$.

- The points $[1 : a_1 : a_2]$ form a plane $\mathbb{C}^2$.
- The points $[0 : a_1 : a_2]$, with $(a_1, a_2) \neq (0,0)$, form a circle (each representing a direction at infinity).

For a set $X \subseteq \mathbb{P}^2$, we let I_X be the ideal of $\mathbb{C}[x_0, x_1, x_2]$ generated by all homogeneous polynomials vanishing at all points of X.

Definition (Coordinate ring of X)

$$\mathbb{C}[X] := \mathbb{C}[x_0, x_1, x_2]/I_X$$

Example

- $I_{\mathbb{P}^2} = \langle 0 \rangle$ and $\mathbb{C}[\mathbb{P}^2] = \mathbb{C}[x_0, x_1, x_2]$
- If $X = \{[1 : 0 : 0]\}$, then $I_X = \langle x_1, x_2 \rangle$ and $\mathbb{C}[X] \cong \mathbb{C}[x_0]$

For $X \subseteq \mathbb{P}^2$, the coordinate ring $\mathbb{C}[X]$ is graded:

$$\mathbb{C}[X] = \bigoplus_{t \in \mathbb{N}} \mathbb{C}[X]_t,$$

where $\mathbb{C}[X]_t$ contains cosets of forms of degree t.

Definition (Hilbert function of X)

$$\mathbf{H}_X(t) := \dim_{\mathbb{C}}(\mathbb{C}[X]_t), \quad t \in \mathbb{N}.$$

Example

- $\mathbf{H}_{\mathbb{P}^2}(t) = \binom{t+2}{2}$
- If $X = \{[1:0:0]\}$, then $\mathbf{H}_X(t) = 1$ for all $t \in \mathbb{N}$.

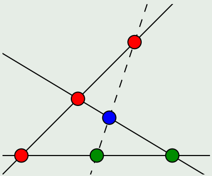
Proposition

Let $X = \{P_1, \ldots, P_n\} \subset \mathbb{P}^2$*.*

- $I_X = I_{P_1} \cap I_{P_2} \cap \cdots \cap I_{P_n}$
- $\mathbf{H}_X(t) = |X| = n$ *for all* $t \gg 0$

Example

Let X be the k-configuration of type $(1, 2, 3)$ depicted below.



t	0	1	2	3	4	5	...
$\mathbf{H}_X(t)$	1	3	6	6	6	6	...

Consider a function $h\colon \mathbb{N} \to \mathbb{N}$.

Problem

Is there $X \subseteq \mathbb{P}^2$ such that $\mathbf{H}_X(t) = h(t)$ for all $t \in \mathbb{N}$?

Macalauy gave an answer in terms of monomial ideals, corresponding to points (with multiplicities) along the axes of $\mathbb{P}^2$.

Problem

Is there a finite set of points $X \subseteq \mathbb{P}^2$ such that $\mathbf{H}_X(t) = h(t)$ for all $t \in \mathbb{N}$?

Given a function $h \colon \mathbb{N} \to \mathbb{N}$, the *first difference* of h is the function $\Delta h \colon \mathbb{N} \to \mathbb{N}$ defined by

$$\Delta h(t) := \begin{cases} h(0), & t = 0, \\ h(t) - h(t-1), & t > 0. \end{cases}$$

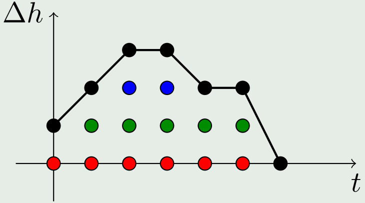
Theorem (Geramita, Maroscia, Roberts 1983)

The function $h \colon \mathbb{N} \to \mathbb{N}$ is the Hilbert function of a finite set of points in $\mathbb{P}^2$ if and only if there exist $\alpha \leqslant \beta \in \mathbb{N}$ such that

1. $\Delta h(t) = t + 1$ *for* $0 \leqslant t < \alpha$,
2. $\Delta h(t) \geqslant \Delta h(t+1)$ *for* $t \geqslant \alpha$,
3. $\Delta h(t) = 0$ *for all* $t \geqslant \beta$.

Example

t	0	1	2	3	4	5	6	7	...
$h(t)$	1	3	6	9	11	13	13	13	...
$\Delta h(t)$	1	2	3	3	2	2	0	0	...



If X is the set of colored points, then $h = \mathbf{H}_X$.
The set X is a k-configuration of type $(2, 5, 6)$.

Theorem (Roberts, Roitman 1989)

All k-configurations of type $(d_1, \ldots, d_s)$ *have the same Hilbert function.*

Finer invariants also fail to distinguish k-configurations of the same type.

Theorem (Geramita, Harima, Shin, 2000)

All k-configurations of type $(d_1, \ldots, d_s)$ *have the same graded Betti numbers.*

Let $X = \{P_1, P_2, \ldots, P_n\} \subset \mathbb{P}^2$.

Definition (Symbolic powers of an ideal of points)

The m-th *symbolic power* of I_X is the ideal

$$I_X^{(m)} := I_{P_1}^m \cap I_{P_2}^m \cap \cdots \cap I_{P_n}^m.$$

Equivalently, $I_X^{(m)}$ is the ideal generated by all homogeneous polynomials that vanish with order at least m at every point of X.

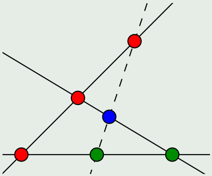
The vanishing locus of $I_X^{(m)}$, denoted mX, consists of all points of X each with multiplicity m. This is an example of a *fat point scheme* supported on X.

Definition (Hilbert function of mX)

$$\mathbf{H}_{mX}(t) := \dim_{\mathbb{C}} \left(\mathbb{C}[x_0, x_1, x_2]/I_X^{(m)} \right)_t$$

Example

Let X be the k-configuration of type $(1, 2, 3)$ depicted below.



t	0	1	2	3	4	5	6	...
$\mathbf{H}_{2X}(t)$	1	3	6	10	14	18	18	...

Our result distinguishes k-configurations of the same type using Hilbert functions and symbolic powers/fat point schemes.

Theorem (G., Shin, Van Tuyl, 2017)

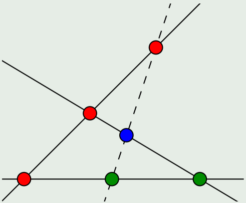
Let $X \subseteq \mathbb{P}^2$ be a k-configuration of type $(d_1, \ldots, d_s) \neq (1)$. Then there exists $m_0 \in \mathbb{N}$ such that for all $m \geqslant m_0$

$$\Delta \mathbf{H}_{mX}(md_s - 1) = \textit{number of lines containing exactly } d_s \textit{ points of } X.$$

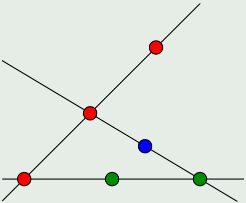
Moreover,

- *if $(d_1, \ldots, d_s) \neq (1, 2, \ldots, s)$, we can take $m_0 = 2$;*
- *if $(d_1, \ldots, d_s) = (1, 2, \ldots, s)$, we can take $m_0 = s + 1$.*

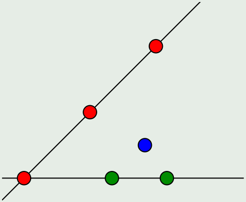
Example (Different k-configurations of type $(1,2,3)$)



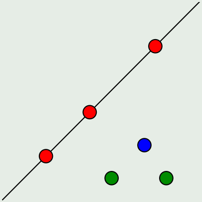
$\mathbf{\Delta H}_{4X}(11) = 4$



$\mathbf{\Delta H}_{4X}(11) = 3$



$\mathbf{\Delta H}_{4X}(11) = 2$



$\mathbf{\Delta H}_{4X}(11) = 1$

Outline of the proof.

- Uniform labeling of the lines through the k-configuration.
- Use a result of Cooper-Harbourne-Teitler (2011) to bound values of the Hilbert function of mX:

$$f_{\mathbf{v}}(t) \leqslant \mathbf{H}_{mX}(t) \leqslant F_{\mathbf{v}}(t)$$

  in terms of a reduction vector $\mathbf{v}$.
- For our k-configurations the bounds coincide, allowing us to relate $\Delta\mathbf{H}_{mX}(t)$ to the number of lines containing the maximum number of points.
- The type $(1, 2, \ldots, s)$ needs further analysis with tools from commutative algebra.

□

Open questions:

- The value $m_0 = s + 1$ for the type $(1, 2, \ldots, s)$ is not optimal. What is the smallest possible value of m_0?
- What other discrete invariants can we use to completely characterize an isomorphism class of k-configurations?
- Can symbolic powers of k-configurations be used to separate components of a Hilbert scheme of points?

If $X \subseteq \mathbb{P}^2$ is a finite set of points, then $\mathbf{H}_{mX}(t) = e$ for all $t \gg 0$.

Definition (Regularity index)

$$\begin{aligned}\mathrm{ri}(mX) &= \min\{t \in \mathbb{N} \mid \mathbf{H}_{mX}(t) = e\} \\ &= \max\{t \in \mathbb{N} \mid \mathbf{\Delta H}_{mX}(t) \neq 0\}\end{aligned}$$

Corollary

Let $X \subseteq \mathbb{P}^2$ be a k-configuration of type $(d_1, \ldots, d_s)$. Then for all integers $m \geqslant s+1$ we have

$$\mathrm{ri}(mX) = md_s - 1.$$

Let $h\colon \mathbb{N} \to \mathbb{N}$ be the Hilbert function of a finite set of points in $\mathbb{P}^2$.

Problem (Geramita, Migliore, Sabourin, 2006)

What are the possible Hilbert functions of fat point schemes whose support has Hilbert function h?

Corollary

Fix integers $m \geqslant s+1 \geqslant 3$. Let X be a k-configuration of type $(1, 2, \ldots, s)$. There are at least $s+1$ possible Hilbert functions of fat points schemes whose support has Hilbert function $\mathbf{H}_X$.