Towards Newton-Okounkov bodies of Hessenberg varieties

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Federico Galetto

joint with H. Abe, L. DeDieu, and M. Harada

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Newton-Okounkov bodies

- $X \subseteq \mathbb{P}^n$, projective variety
- $\nu \colon \mathbb{C}[X] \setminus \{0\} \to \mathbb{Z}^n$, valuation

Definition

The *Newton-Okounkov body* of $X \subseteq \mathbb{P}^n$ is

$$\Delta(X,\nu) := \overline{\operatorname{conv}\left(\bigcup_{d>0} \left\{ \frac{\nu(f)}{d} \,\middle|\, f \in \mathbb{C}[X]_d \setminus \{0\} \right\}\right)} \subseteq \mathbb{R}^n.$$

- $\dim \Delta(X,\nu) = \dim X =: d$
- $\operatorname{vol} \Delta(X,\nu) = \frac{1}{d!} \deg X$

Hessenberg varieties

Denote $V_\bullet \in \mathrm{Flags}(\mathbb{C}^n)$ the point corresponding to

$$0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset V_n = \mathbb{C}^n.$$

Definition

Given

- A, $n \times n$ complex matrix
- $h\colon \{1, \dots, n\} \to \{1, \dots, n\}$, weakly increasing with $h(i) \geqslant i$

the *Hessenberg variety* associated to A and h is

$$\mathrm{Hess}(A, h) := \{V_\bullet \in \mathrm{Flags}(\mathbb{C}^n) \mid AV_i \subseteq V_{h(i)}\}.$$

Regular nilpotent Hessenberg varieties

Example

When

$$A = N := \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix},$$

$\mathrm{Hess}(N, h) = \{V_\bullet \mid NV_i \subseteq V_{h(i)}\}$ is called *regular nilpotent*.

Example

If $A = N$ and

$$h(i) = \begin{cases} i+1 & i < n \\ n & i = n, \end{cases}$$

then $\mathrm{Pet}_n := \mathrm{Hess}(N, h)$ is the *Peterson variety*.

Newton-Okounkov bodies of Peterson varieties

- $\lambda = (a_1 + a_2, a_1, 0)$, dominant weight
- $\mathrm{Pet}_3 \subset \mathrm{Flags}(\mathbb{C}^3) \hookrightarrow \mathbb{P}(V_\lambda^*)$, Plücker embedding
- ν, "order of vanishing" along a chain of subvarieties

Theorem (ADGH)

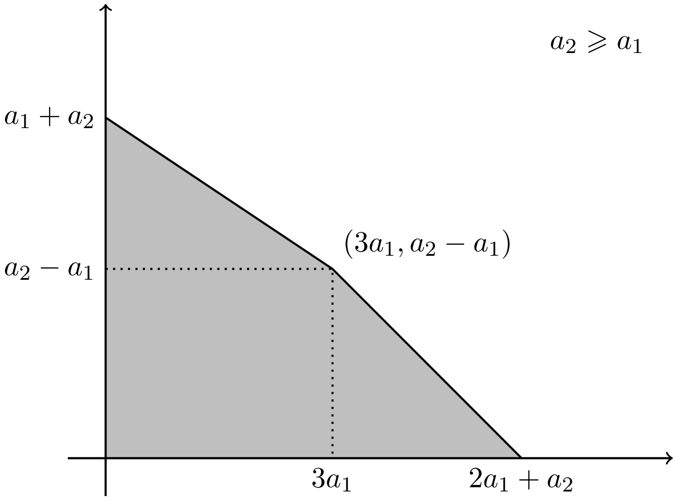
If $a_2 \geqslant a_1$, then $\Delta(\mathrm{Pet}_3, \nu)$ is

$$\mathrm{conv}\{(0,0), (2a_1 + a_2, 0), (0, a_1 + a_2), (3a_1, a_2 - a_1)\}.$$

If $a_2 < a_1$, then $\Delta(\mathrm{Pet}_3, \nu)$ is

$$\mathrm{conv}\{(0,0), (2a_2 + a_1, 0), (0, a_1 + a_2), (3a_2, a_1 - a_2)\}.$$

Newton-Okounkov bodies of Peterson varieties



A family of Hessenberg varieties

- $\lambda_1, \ldots, \lambda_n$ distinct complex numbers
- $\Lambda_t := \begin{bmatrix} t\lambda_1 & 1 & & & \\ & t\lambda_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & t\lambda_{n-1} & 1 \\ & & & & t\lambda_n \end{bmatrix}$
- $\mathfrak{X}_h := \{(V_\bullet, t) \in \mathrm{Flags}(\mathbb{C}^n) \times \mathbb{C} \mid \Lambda_t V_i \subseteq V_{h(i)}\}$

D. Anderson and J. Tymoczko show that $\mathfrak{X}_h \to \mathbb{C}$ is a flat family.

Theorem (ADGH)

The fibers over the closed points of $\mathfrak{X}_h \to \mathbb{C}$ are reduced.

The special fiber is a regular nilpotent Hessenberg variety.

Regular semisimple Hessenberg varieties

Example

If $\lambda_1, \ldots, \lambda_n$ are distinct complex numbers and

$$A = S := \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix},$$

$\mathrm{Hess}(S,h) = \{V_\bullet \mid SV_i \subseteq V_{h(i)}\}$ is called *regular semisimple*.

- The general fiber of $\mathfrak{X}_h \to \mathbb{C}$ is a regular semisimple Hessenberg variety.
- Regular semisimple Hessenberg varieties are smooth.

Degree of regular semisimple Hessenberg

- $\mathrm{Hess}(S,h) \subset \mathrm{Flags}(\mathbb{C}^n) \hookrightarrow \mathbb{P}(V_\lambda^*)$, Plücker embedding
- $d = \dim \mathrm{Hess}(S,h)$
- [T. Abe, T. Horiguchi, M. Masuda, S. Murai and T. Sato]

$$P_h(x_1,\ldots,x_n) := \left(\prod_{h(i)<j} \partial_j - \partial_i\right) \prod_{1\leqslant k<l\leqslant n} \frac{x_k - x_l}{l-k},$$

volume polynomial

Using methods of symplectic geometry, we show that

$$\deg \mathrm{Hess}(S,h) = d! P_h(\lambda_1,\ldots,\lambda_n).$$

Degree of regular nilpotent Hessenberg

- $\mathfrak{X}_h \to \mathbb{C}$, flat family with reduced fibers
- $\mathrm{Hess}(S, h)$, general fiber
- $\mathrm{Hess}(N, h)$, special fiber
- degree is preserved along flat families

Theorem (ADGH)

$$\deg \mathrm{Hess}(N, h) = d! P_h(\lambda_1, \dots, \lambda_n)$$

Wait, there's more!

Along the way we:

- describe local equations for $\mathrm{Hess}(N,h)$ and $\mathfrak{X}_h$;
- prove that all regular nilpotent Hessenberg varieties are local complete intersections (hence Cohen-Macaulay and Gorenstein), generalizing work of E. Insko and A. Yong;
- describe intersections of $\mathrm{Hess}(N,h)$ with certain Schubert varieties;
- construct chains of subvarieties in $\mathrm{Hess}(N,h)$ which give rise to nice geometric valuations.