Betti numbers of symbolic powers of star configurations

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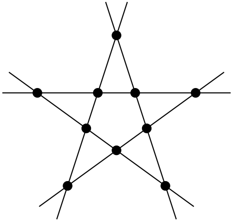
Star configurations

- $L_1, \ldots, L_n$ linear forms in a polynomial ring
- Assume all subsets $\{L_{i_1}, \ldots, L_{i_c}\}$ are linearly independent

Definition (Star configuration of codimension c)

$$I_{n,c} := \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \langle L_{i_1}, \ldots, L_{i_c} \rangle$$



Symbolic powers of star configurations

For all $m \geqslant 1$,

$$I_{n,c}^{(m)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \langle L_{i_1}, \ldots, L_{i_c} \rangle^m.$$

Problem

Can we describe the Betti numbers of $I_{n,c}^{(m)}$?

Bonus problem

Can we describe the equivariant Betti numbers of $I_{n,c}^{(m)}$?

Known: Betti numbers of symbolic square

Theorem (Geramita, Harbourne, Migliore, 2013)

If $c \geqslant 2$, then

$$\beta_{i,i+j}(I_{n,c}^{(2)}) = \begin{cases} \binom{n}{c-2-i}\binom{n-c+1+i}{i}, & j = n-c+2 \\ \binom{n}{c-1}\binom{c-1}{i}, & j = 2(n-c+1) \end{cases}$$

G., Geramita, Shin, and Van Tuyl also prove this for codimension 2 star configurations via symbolic defect.

Reduction to monomials

Theorem (Geramita, Harbourne, Migliore, Nagel, 2017)

If we replace the linear forms L_i by variables x_i, then the Betti numbers of $I_{n,c}^{(m)}$ stay the same.

From now on, we consider

$$I_{n,c}^{(m)} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant n} \langle x_{i_1}, \ldots, x_{i_c} \rangle^m \subseteq \Bbbk[x_1, \ldots, x_n].$$

Advantages:

- $I_{n,c}^{(m)}$ is a monomial ideal;
- $I_{n,c}^{(m)}$ is stable under permutations of variables.

$\mathfrak{S}_n$-fixed ideals

The symmetric group $\mathfrak{S}_n$ acts on $\Bbbk[x_1, \dots, x_n]$ by permuting the variables.
Let $I \subseteq \Bbbk[x_1, \dots, x_n]$ be a monomial ideal such that $\mathfrak{S}_n \cdot I \subseteq I$.
The minimal generating set $G(I)$ of I splits into $\mathfrak{S}_n$-orbits:

$$\{\sigma(x^\lambda) : \sigma \in \mathfrak{S}_n\}$$

for some partitions $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$.
[Convention: partitions have $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$.]

Definition

For an $\mathfrak{S}_n$-fixed monomial ideal $I \subseteq \Bbbk[x_1, \dots, x_n]$, define

$$P(I) := \{\lambda : x^\lambda \in I\},$$
$$\Lambda(I) := \{\lambda : x^\lambda \in G(I)\}.$$

Shifted ideals

Let $I \subset \mathbb{k}[x_1, \dots, x_n]$ be an $\mathfrak{S}_n$-fixed monomial ideal.

Definition (Shifted ideal)

We say I is *shifted* if, for every $\lambda = (\lambda_1, \dots, \lambda_n) \in P(I)$ and $1 \leqslant k < n$ with $\lambda_k < \lambda_n$, we have $x^\lambda x_k / x_n \in I$.

Definition (Strongly shifted ideal)

We say I is *strongly shifted* if, for every $\lambda = (\lambda_1, \dots, \lambda_n) \in P(I)$ and $1 \leqslant k < l \leqslant n$ with $\lambda_k < \lambda_l$, we have $x^\lambda x_k / x_l \in I$.

In both definitions, we can replace $P(I)$ by $\Lambda(I)$.

Examples of shifted ideals

Example

The $\mathfrak{S}_3$-fixed ideal

$$\begin{aligned} I = \langle & x_1x_2x_3, \\ & x_1^2x_2, x_1x_2^2, x_1^2x_3, x_1x_3^2, x_2^2x_3, x_2x_3^2, \\ & x_1^4, x_2^4, x_3^4\rangle \subseteq \Bbbk[x_1, x_2, x_3] \end{aligned}$$

is strongly shifted with $\Lambda(I) = \{(1,1,1), (0,1,2), (0,0,4)\}$.

Example

The $\mathfrak{S}_4$-fixed ideal $I \subseteq \Bbbk[x_1, x_2, x_3, x_4]$ with $\Lambda(I) = \{(1,1,2,2), (0,2,2,2), (0,1,2,3)\}$ is shifted but not strongly shifted since $(0,1,2,3) \in P(I)$ but $(1,1,1,3) \notin P(I)$.

Star configurations are strongly shifted

Proposition (BDGMNORS)

For every integer $m \geqslant 1$, $I_{n,c}^{(m)}$ is $\mathfrak{S}_n$-fixed and strongly shifted. Moreover

$$P(I_{n,c}^{(m)}) = \left\{ \lambda : \sum_{i=1}^{c} \lambda_i \geqslant m \right\},$$

$$\Lambda(I_{n,c}^{(m)}) = \left\{ \lambda : \sum_{i=1}^{c} \lambda_i = m, \forall i > c \ \lambda_i = \lambda_c \right\}.$$

Question

Are there other interesting examples of (strongly) shifted ideals?

Shifted ideals have linear quotients

Consider distinct monomials $u = \sigma(x^\lambda), v = \tau(x^\mu) \in \Bbbk[x_1, \dots, x_n]$, where λ, μ are partitions, and $\sigma, \tau \in \mathfrak{S}_n$.
We set $v \prec u$ if:

- $\deg(v) < \deg(u)$, or
- $\deg(v) = \deg(u)$ and $x^\mu >_{\text{lex}} x^\lambda$, or
- $\lambda = \mu$ and $v <_{\text{lex}} u$.

Theorem (BDGMNORS)

Shifted $\mathfrak{S}_n$-fixed monomial ideals have linear quotients.

Betti tables of star configurations

Corollary (BDGMNORS)

1. *For every integer* $i \geqslant 0$,

$$\beta_{i,i+m(n-c+1)}(I_{n,c}^{(m)}) = \binom{n}{c-1}\binom{c-1}{i}.$$

2. *The Castelnuovo-Mumford regularity of* $I_{n,c}^{(m)}$ *is* $m(n-c+1)$.
3. *If* $m \geqslant 2$, *then all nonzero rows in the Betti table of* $I_{n,c}^{(m)}$ *have length* $c-1$, *with the exception of the top one.*
4. *If* $m \leqslant c$, *then for every integer* $i \geqslant 0$,

$$\beta_{i,i+n-c+m}(I_{n,c}^{(m)}) = \binom{n}{c-m-i}\binom{n-c+m+i-1}{i}.$$

Betti numbers of symbolic cube

Corollary (BDGMNORS)

If $c \geqslant 3$, *then* $\beta_{i,i+j}(I_{n,c}^{(3)}) =$

$$\begin{cases} \binom{n}{c-3-i}\binom{n-c+2+i}{i}, & j = n-c+3 \\ \binom{n}{c-2}\left(\binom{c-2}{i} + (n-c+1)\binom{c-1}{i}\right), & j = 2(n-c+1) \\ \binom{n}{c-1}\binom{c-1}{i}, & j = 3(n-c+1) \end{cases}$$