The symbolic defect of an ideal

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Federico Galetto
joint with A.V. Geramita, Y.S. Shin, and A. Van Tuyl

McMaster University

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Symbolic powers

Let $R = \Bbbk[x_0, x_1, \ldots, x_n]$ and let $I \subseteq R$ be an ideal.

Definition (m-th symbolic power)

$$I^{(m)} = \bigcap_{P \in \operatorname{Ass}(I)} (I^m R_P \cap R)$$

We can think of elements of $I^{(m)}$ as polynomials vanishing on the zero locus of I in $\mathbb{P}^n$ with multiplicity at least m.

Observation

$$I^m \subseteq I^{(m)}$$

Containment and comparison

Containment problem

Given m, find the smallest r such that

$$I^{(r)} \subseteq I^m.$$

Comparison problem

Given m, how far is I^m from $I^{(m)}$? Or how big is $I^{(m)}/I^m$?

Some work related to the second problem:

- Arsie-Vatne (saturation)
- Huneke, Herzog, Ulrich, and Vasconcelos (height two primes)
- Schenzel (monomial curves)

Symbolic defect

Definition (Symbolic defect of an ideal I)

$$\operatorname{sdefect}(I, m) := \mu\big(I^{(m)}/I^m\big)$$

where μ is the minimal number of generators

Observations

- $\operatorname{sdefect}(I, m) = t \Rightarrow I^{(m)} = I^m + \langle F_1, \ldots, F_t \rangle$
- $\operatorname{sdefect}(I, 1) = 0$
- I complete intersection $\Rightarrow \operatorname{sdefect}(I, m) = 0$ for all m

The first non-trivial case is when $\operatorname{sdefect}(I, 2) = 1$.

Symbolic defect and points

Let $X \subseteq \mathbb{P}^2$ be a finite set of points, I_X its defining ideal.

Theorem

The following are equivalent

1. I_X is a complete intersection
2. $\operatorname{sdefect}(I_X, m) = 0$ for all $m \geqslant 1$
3. $\operatorname{sdefect}(I_X, m) = 0$ for some $m \geqslant 2$

Symbolic defect and general points

Theorem (G., Geramita, Shin, Van Tuyl, 2016)

For $X \subseteq \mathbb{P}^2$ a general set of s points:

1. $\text{sdefect}(I_X, 2) = 0$ if and only if $s = 1, 2, 4$
2. $\text{sdefect}(I_X, 2) = 1$ if and only if $s = 3, 5, 7, 8$
3. $\text{sdefect}(I_X, 2) > 1$ if and only if $s = 6$ or $s \geqslant 9$

In the proof, we use

- Alexander-Hirschowitz theorem on the Hilbert function of $I_X^{(2)}$;
- works of Catalisano, Geramita, Gregory, Harbourne, Idà, Lorenzini, Maroscia, and Roberts to resolve I_X and $I_X^{(2)}$;
- Hilbert series computations.

Symbolic defect sequence

Definition (Symbolic defect sequence of I)

$$\{\mathrm{sdefect}(I,m)\}_{m=0}^{\infty}$$

Question

What can be said about the symbolic defect sequence?

Example

For $X \subseteq \mathbb{P}^2$ a set of 8 general points:

$$\{\mathrm{sdefect}(I,m)\}_{m=0}^{\infty} = 0 \quad 1 \quad 3 \quad 6 \quad 10 \quad 9 \quad 7$$

Star configurations

Let $\mathcal{F} = \{F_1, \ldots, F_s\}$ be a set of forms in $\Bbbk[x_0, x_1, \ldots, x_n]$ such that all subsets of $\mathcal{F}$ of cardinality $c+1$ are regular sequences.

Definition (Star configuration)

$$I_{c,\mathcal{F}} = \bigcap_{1 \leqslant i_1 < \cdots < i_c \leqslant s} \langle F_{i_1}, \ldots, F_{i_c} \rangle$$

The zero locus of $I_{c,\mathcal{F}}$ in $\mathbb{P}^n$ is called a star configuration.

Definition (Linear star configuration)

If $F_1, \ldots, F_s$ are linear forms, the star configuration is called linear.

Symbolic defect and star configurations

Theorem (G., Geramita, Shin, Van Tuyl, 2016)

- $\operatorname{sdefect}(I_{c,\mathcal{F}}, 2) \leqslant \binom{s}{c-2}$, with equality in the linear case
- $\operatorname{sdefect}(I_{c,\mathcal{F}}, 3) \leqslant \binom{s}{c-3} + \binom{s}{c-2}\binom{s}{c-1}$
- $\operatorname{sdefect}(I_{c,\mathcal{F}}, m) = 1$ if and only if $c = m = 2$

Using a result of Geramita-Harbourne-Migliore-Nagel, we reduce to the monomial case, where we perform direct computations.

Symbolic defect forcing geometry

As a partial converse of the previous theorem, we prove:

Theorem (G., Geramita, Shin, Van Tuyl, 2016)

Let X be a set of $\binom{\alpha+1}{2}$ points in $\mathbb{P}^2$ with generic Hilbert function. If $\operatorname{sdefect}(I_X, 2) = 1$, then X is a linear star configuration.

Our proof uses a theorem of Bocci-Chiantini and degree considerations on the generators of I_X and $I_X^{(2)}$.

Symbolic defect and edge ideals

Theorem (Janssen, Kamp, Vander Woude, 2017)

If I is the edge ideal of a cycle of length $2n+1$, then

- $\operatorname{sdefect}(I,m) = 0$ for $1 \leqslant m \leqslant n$;
- $\operatorname{sdefect}(I,n+1) = 1$;
- for $n+2 \leqslant m \leqslant 2n+1$

$$\operatorname{sdefect}(I,m) = \sum_{i=1}^{2n+1} \binom{2n+1}{i} \binom{i}{m-n-1-i}.$$

Symbolic defect and cover ideals of graphs

Theorem (Drabkin, Guerrieri, 2018)

If the symbolic Rees algebra of I is Noetherian, then $\operatorname{sdefect}(I, m)$ is quasi-polynomial.

Drabkin and Guerrieri also prove several statements on the symbolic defect of cover ideal of graphs, such as

Theorem (Drabkin, Guerrieri, 2018)

Let I be the cover ideal of a graph G. We have $\operatorname{sdefect}(I, 2) = 1$ if and only if G is non-bipartite and every vertex of G is adjacent to every odd cycle in G.