The symbolic defect of an ideal
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Symbolic powers

Let \( R = \mathbb{k}[x_0, x_1, \ldots, x_n] \) and let \( I \subseteq R \) be an ideal.

**Definition \((m\text{-th symbolic power})\)**

\[
I^{(m)} = \bigcap_{P \in \text{Ass}(I)} (I^m R_P \cap R)
\]

We can think of elements of \( I^{(m)} \) as polynomials vanishing on the zero locus of \( I \) in \( \mathbb{P}^n \) with multiplicity at least \( m \).

**Observation**

\[
I^m \subseteq I^{(m)}
\]
Containment and comparison

Containment problem
Given $m$, find the smallest $r$ such that
\[ I^{(r)} \subseteq I^m. \]

Comparison problem
Given $m$, how far is $I^m$ from $I^{(m)}$? Or how big is $I^{(m)}/I^m$?

Some work related to the second problem:
- Arsie-Vatne (saturation)
- Huneke, Herzog, Ulrich, and Vasconcelos (height two primes)
- Schenzel (monomial curves)
Symbolic defect

Definition (Symbolic defect of an ideal $I$)

\[ \text{sdefect}(I, m) := \mu\left(\frac{I^{(m)}}{I^m}\right) \]

where $\mu$ is the minimal number of generators

Observations

- $\text{sdefect}(I, m) = t \Rightarrow I^{(m)} = I^m + \langle F_1, \ldots, F_t \rangle$
- $\text{sdefect}(I, 1) = 0$
- $I$ complete intersection $\Rightarrow \text{sdefect}(I, m) = 0$ for all $m$

The first non-trivial case is when $\text{sdefect}(I, 2) = 1$. 
Let $X \subseteq \mathbb{P}^2$ be a finite set of points, $I_X$ its defining ideal.

**Theorem**

The following are equivalent

1. $I_X$ is a complete intersection
2. $s\text{defect}(I_X, m) = 0$ for all $m \geq 1$
3. $s\text{defect}(I_X, m) = 0$ for some $m \geq 2$
Symbolic defect and general points

**Theorem (G., Geramita, Shin, Van Tuyl, 2016)**

For $X \subseteq \mathbb{P}^2$ a general set of $s$ points:

1. $s\text{defect}(I_X, 2) = 0$ if and only if $s = 1, 2, 4$
2. $s\text{defect}(I_X, 2) = 1$ if and only if $s = 3, 5, 7, 8$
3. $s\text{defect}(I_X, 2) > 1$ if and only if $s = 6$ or $s \geq 9$

In the proof, we use

- Alexander-Hirschowitz theorem on the Hilbert function of $I_X^{(2)}$;
- works of Catalisano, Geramita, Gregory, Harbourne, Idà, Lorenzini, Maroscia, and Roberts to resolve $I_X$ and $I_X^{(2)}$;
- Hilbert series computations.

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Symbolic defect sequence

**Definition (Symbolic defect sequence of $I$)**

\[ \{s\text{defect}(I, m)\}_{m=0}^{\infty} \]

**Question**

What can be said about the symbolic defect sequence?

**Example**

For $X \subseteq \mathbb{P}^2$ a set of 8 general points:

\[ \{s\text{defect}(I, m)\}_{m=0}^{\infty} = 0 \ 1 \ 3 \ 6 \ 10 \ 9 \ 7 \]
Let $\mathcal{F} = \{F_1, \ldots, F_s\}$ be a set of forms in $\mathbb{k}[x_0, x_1, \ldots, x_n]$ such that all subsets of $\mathcal{F}$ of cardinality $c + 1$ are regular sequences.

**Definition (Star configuration)**

$$I_{c,\mathcal{F}} = \bigcap_{1 \leq i_1 < \cdots < i_c \leq s} \langle F_{i_1}, \ldots, F_{i_c} \rangle$$

The zero locus of $I_{c,\mathcal{F}}$ in $\mathbb{P}^n$ is called a star configuration.

**Definition (Linear star configuration)**

If $F_1, \ldots, F_s$ are linear forms, the star configuration is called linear.
Symbolic defect and star configurations

Theorem (G., Geramita, Shin, Van Tuyl, 2016)

- $s\text{defect}(I_{c,\mathcal{F}}, 2) \leq \binom{s}{c-2}$, with equality in the linear case
- $s\text{defect}(I_{c,\mathcal{F}}, 3) \leq \binom{s}{c-3} + \binom{s}{c-2} \binom{s}{c-1}$
- $s\text{defect}(I_{c,\mathcal{F}}, m) = 1$ if and only if $c = m = 2$

Using a result of Geramita-Harbourne-Migliore-Nagel, we reduce to the monomial case, where we perform direct computations.
As a partial converse of the previous theorem, we prove:

**Theorem (G., Geramita, Shin, Van Tuyl, 2016)**

Let $X$ be a set of $\binom{\alpha+1}{2}$ points in $\mathbb{P}^2$ with generic Hilbert function. If $\text{sdefect}(I_X, 2) = 1$, then $X$ is a linear star configuration.

Our proof uses a theorem of Bocci-Chiantini and degree considerations on the generators of $I_X$ and $I_X^{(2)}$. 

Symbolic defect forcing geometry
Theorem (Janssen, Kamp, Vander Woude, 2017)

If $I$ is the edge ideal of a cycle of length $2n + 1$, then

1. $s\text{defect}(I, m) = 0$ for $1 \leq m \leq n$;
2. $s\text{defect}(I, n + 1) = 1$;
3. for $n + 2 \leq m \leq 2n + 1$

$$s\text{defect}(I, m) = \sum_{i=1}^{2n+1} \binom{2n + 1}{i} \binom{i}{m - n - 1 - i}.$$
Symbolic defect and cover ideals of graphs

**Theorem (Drabkin, Guerrieri, 2018)**

If the symbolic Rees algebra of $I$ is Noetherian, then $s\text{defect}(I, m)$ is quasi-polynomial.

Drabkin and Guerrieri also prove several statements on the symbolic defect of cover ideal of graphs, such as

**Theorem (Drabkin, Guerrieri, 2018)**

Let $I$ be the cover ideal of a graph $G$. We have $s\text{defect}(I, 2) = 1$ if and only if $G$ is non-bipartite and every vertex of $G$ is adjacent to every odd cycle in $G$. 