Jets of graphs

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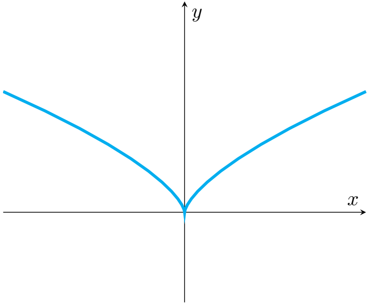
Overview

Algebraic geometry

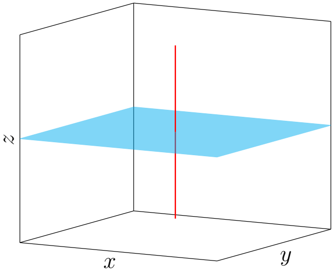


Commutative algebra

$$y^3 = x^2$$

Overview

Algebraic geometry

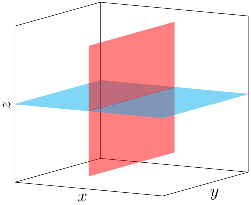


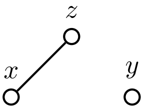
Commutative algebra

$$\begin{cases} xz = 0 \\ yz = 0 \end{cases}$$

Monomial equations and graphs

$$xz = 0$$





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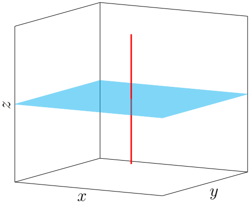

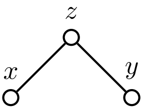

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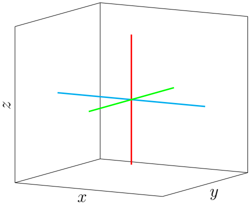
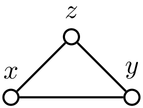
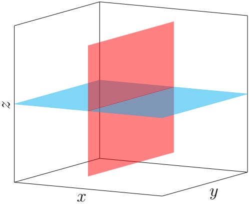


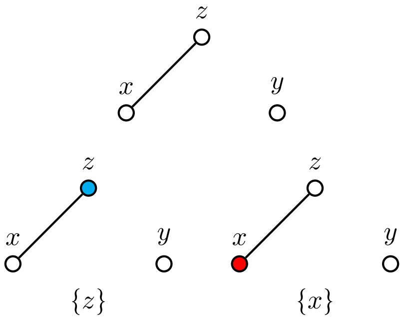


Monomial equations and graphs

A *minimal vertex cover* of a graph is a minimal set of vertices that contains an endpoint of every edge of the graph.
Vertex covers correpsond to geometric components.

$$xz = 0$$





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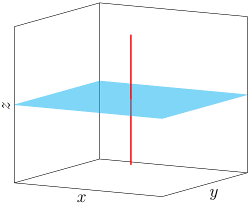

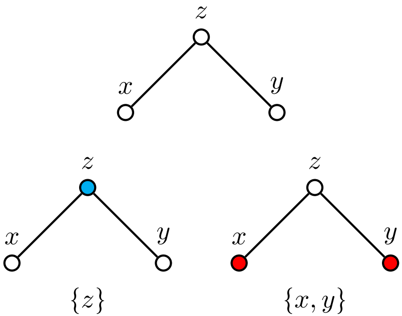

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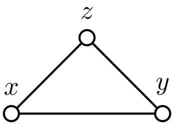
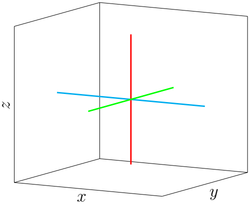
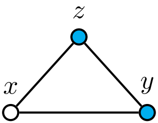
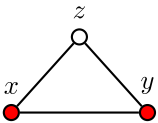


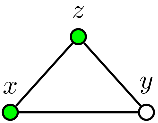




$\{y, z\}$

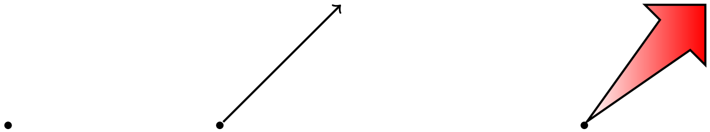


$\{x, y\}$



$\{x, z\}$

Jets and jet equations



0-jet: point — (x_0, y_0)

1-jet: point + velocity — $(x_0 + x_1 t, y_0 + y_1 t)$

2-jet: point + velocity + acceleration — $\left(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2\right)$

A 2D jet of order s:

$$\left(x_0 + x_1 t + x_2 t^2 + \cdots + x_s t^s, y_0 + y_1 t + y_2 t^2 + \cdots + y_s t^s\right)$$

Jets and jet equations

The s-jets of a curve $f(x, y) = 0$ must satify the equation:

$$f\left(x_0 + x_1 t + x_2 t^2 + \cdots + x_s t^s, y_0 + y_1 t + y_2 t^2 + \cdots + y_s t^s\right) = 0$$

When f is a polynomial, the equation becomes

$$f_0 + f_1 t + f_2 t^2 + \cdots + f_s t^s + \cdots = 0 \iff f_0 = f_1 = \cdots = f_s = 0$$

where $f_0, f_1, \ldots, f_s$ are polynomials in the variables x_i, y_i.

Example

$$\begin{aligned} y^3 - x^2 = 0 &\rightsquigarrow (y_0 + y_1 t + y_2 t^2)^3 - (x_0 + x_1 t + x_2 t^2)^2 = 0 \\ &\rightsquigarrow (y_0^3 - x_0^2) + (3y_0^2 y_1 - 2x_0 x_1)t \\ &\quad + (3y_0^2 y_2 + 3y_0 y_1^2 - 2x_0 x_2 - x_1^2)t^2 + \cdots = 0 \end{aligned}$$

Jets and arcs (the infinite counterpart of jets) are especially important within algebraic geometry for their applications to the study of singularities. The use of jets to study singularities was initiated by John Nash in 1995.

Unfortunately, jets of monomials are not necessarily monomials.

Example

$$\begin{aligned} xy = 0 &\rightsquigarrow (x_0 + x_1 t)(y_0 + y_1 t) = 0 \\ &\rightsquigarrow x_0 y_0 + (x_0 y_1 + x_1 y_0)t + \cdots = 0 \end{aligned}$$

However, these jets have the same solutions as a system of monomials.

Example

$$\begin{cases} x_0 y_0 = 0 \\ x_0 y_1 + x_1 y_0 = 0 \end{cases} \iff \begin{cases} x_0 y_0 = 0 \\ x_0 y_1 = 0 \\ x_1 y_0 = 0 \end{cases}$$

Jets of graphs

Theorem (Goward, Smith, 2006)

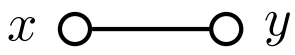
The jet equations of a system of monomials have the same solutions as the system of monomials of the jet equations.

Definition (G., Helmick, Walsh, 2020)

1. Start from a graph G.
2. Construct a system of monomial equations from G.
3. Take the jets of order s of the system.
4. Form a graph from the monomials in the jet equations.

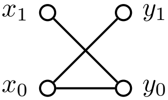
The resulting graph $\mathcal{J}_s(G)$ is the graph of s-jets of G.

G



xy

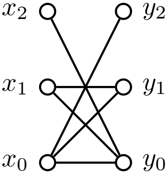
$\mathcal{J}_1(G)$



$(x_0 + x_1 t)(y_0 + y_1 t)$

$x_0 y_0 + (x_0 y_1 + x_1 y_0) t$

$\mathcal{J}_2(G)$



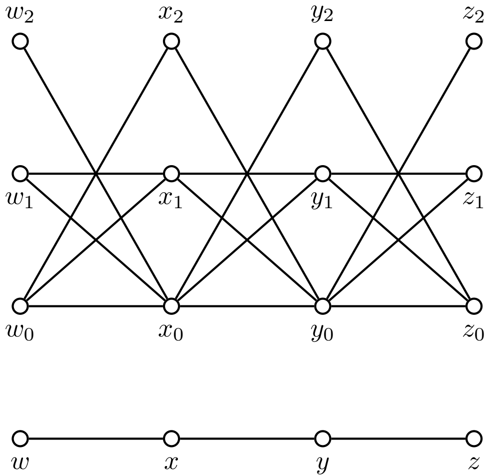
$(x_0 + x_1 t + x_2 t^2)(y_0 + y_1 t + y_2 t^2)$

$x_0 y_0 + (x_0 y_1 + x_1 y_0) t + (x_0 y_2 + x_1 y_1 x_2 y_0) t^2$

Jets of graphs

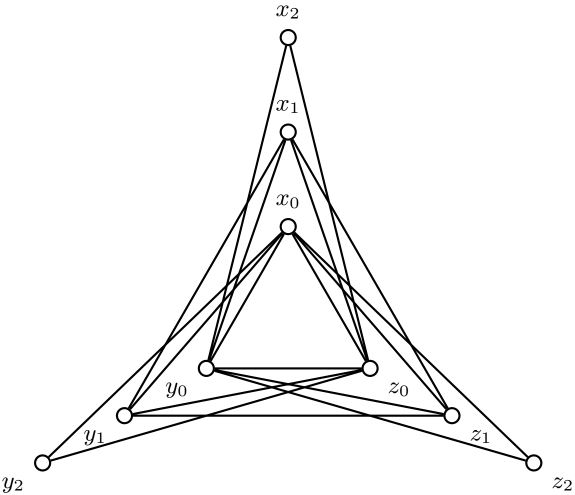
$\mathcal{J}_2(G)$

G



Jets of graphs

$\mathcal{J}_2(G)$

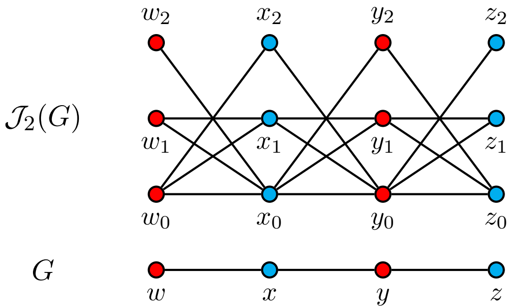


Chromatic numbers of jets

The *chromatic number* is the smallest number of colors needed to color a graph's vertices so that adjacent vertices have different colors.

Theorem (G., Helmick, Walsh, 2020)

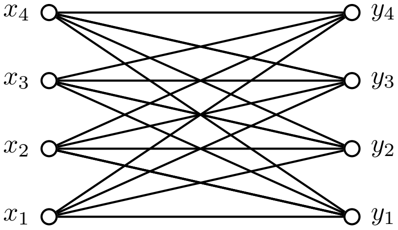
$\mathcal{J}_s(G)$ has the same chromatic number as G.



Vertex covers of jets

A graph is *very well covered* if all its minimal vertex covers contain exactly half of its vertices. The complete bipartite graphs $K_{n,n}$ are very well covered with 2 minimal vertex covers.

$K_{4,4}$



$\{x_1, x_2, x_3, x_4\}$

$\{y_1, y_2, y_3, y_4\}$

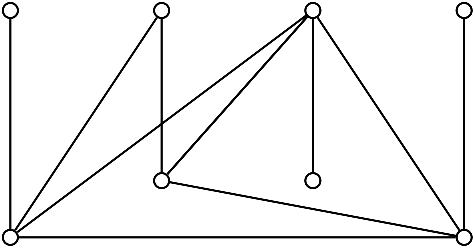
Theorem (G., Helmick, Walsh, 2020)

$\mathcal{J}_s(K_{n,n})$ *is very well covered with* $s+2$ *minimal vertex covers.*

Vertex covers of jets

Conjecture (G., Helmick, Walsh, 2020)

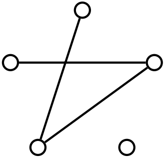
The jets of a very well covered graph are very well covered.



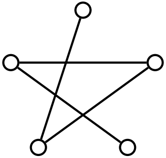
Favaron's very well covered graph G_1

Co-chordal jets

A graph G is *co-chordal* if every cycle of length 4 or more in the complement of G has a chord.



co-chordal



not co-chordal

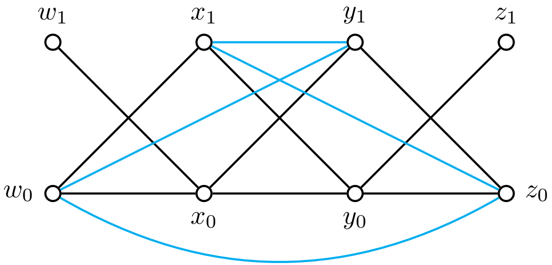
Theorem (Fröberg, 1990)

G is co-chordal if and only if the minimal relations among the monomials associated to the edges of G are linear.

Co-chordal jets

Theorem (G., Helmick, Walsh, 2020)

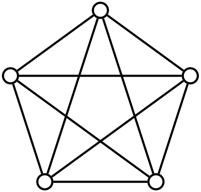
If a graph contains a path of length 3 or more, then its jets are not co-chordal.



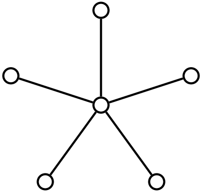
Co-chordal jets

Theorem (G., Helmick, Walsh, 2020)

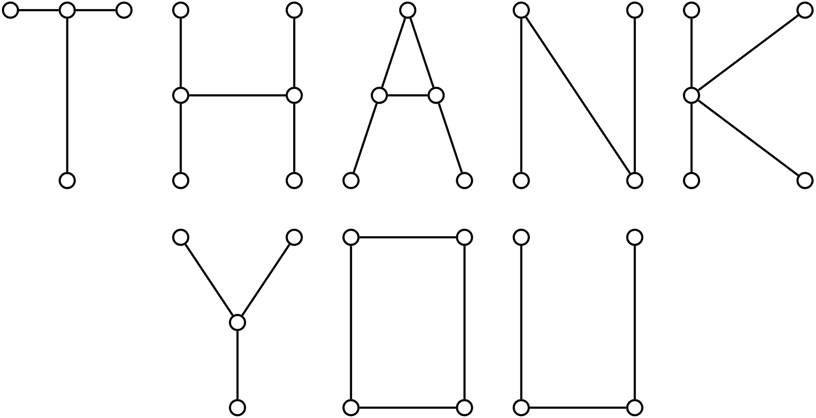
The jets of complete graphs K_n and star graphs $K_{1,n}$ are co-chordal.



K_5



$K_{1,5}$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