Jets of graphs

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Colloquium - April 5, 2021
Overview

Algebraic geometry

Commutative algebra

\[ y^3 = x^2 \]
Overview

Algebraic geometry

Commutative algebra

\[
\begin{aligned}
    & xz = 0 \\
    & yz = 0
\end{aligned}
\]
Monomial equations and graphs

\[ xz = 0 \]

\[ \begin{cases} xz = 0 \\ yz = 0 \end{cases} \]

\[ \begin{cases} xy = 0 \\ xz = 0 \\ yz = 0 \end{cases} \]
A minimal vertex cover of a graph is a minimal set of vertices that contains an endpoint of every edge of the graph. Vertex covers correspond to geometric components.
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\[
\begin{align*}
xy &= 0 \\
xz &= 0 \\
yz &= 0
\end{align*}
\]
Jets and jet equations

0-jet: point
\[(x_0, y_0)\]

1-jet: point + velocity
\[(x_0 + x_1t, y_0 + y_1t)\]

2-jet: point + velocity + acceleration
\[(x_0 + x_1t + x_2t^2, y_0 + y_1t + y_2t^2)\]

A 2D jet of order \(s\):
\[(x_0 + x_1t + x_2t^2 + \cdots + x_st^s, y_0 + y_1t + y_2t^2 + \cdots + y_st^s)\]
The $s$-jets of a curve $f(x, y) = 0$ must satisfy the equation:

$$f \left( x_0 + x_1 t + x_2 t^2 + \cdots + x_st^s, y_0 + y_1 t + y_2 t^2 + \cdots + y_st^s \right) = 0$$

When $f$ is a polynomial, the equation becomes

$$f_0 + f_1 t + f_2 t^2 + \cdots + f_st^s + \cdots = 0 \iff f_0 = f_1 = \cdots = f_s = 0$$

where $f_0, f_1, \ldots, f_s$ are polynomials in the variables $x_i, y_i$.

**Example**

$$y^3 - x^2 = 0 \implies (y_0 + y_1 t + y_2 t^2)^3 - (x_0 + x_1 t + x_2 t^2)^2 = 0$$

$$\implies (y_0^3 - x_0^2) + (3y_0^2y_1 - 2x_0x_1)t$$

$$+ (3y_0^2y_2 + 3y_0y_1^2 - 2x_0x_2 - x_1^2)t^2 + \cdots = 0$$
Jets and arcs (the infinite counterpart of jets) are especially important within algebraic geometry for their applications to the study of singularities. The use of jets to study singularities was initiated by John Nash in 1995.
Unfortunately, jets of monomials are not necessarily monomials.

**Example**

\[ xy = 0 \implies (x_0 + x_1 t)(y_0 + y_1 t) = 0 \]
\[ \implies x_0 y_0 + (x_0 y_1 + x_1 y_0) t + \cdots = 0 \]

However, these jets have the same solutions as a system of monomials.

**Example**

\[
\begin{align*}
x_0 y_0 &= 0 \\
x_0 y_1 + x_1 y_0 &= 0
\end{align*}
\]

\[
\begin{align*}
x_0 y_0 &= 0 \\
x_0 y_1 &= 0 \\
x_1 y_0 &= 0
\end{align*}
\]
Theorem (Goward, Smith, 2006)

*The jet equations of a system of monomials have the same solutions as the system of monomials of the jet equations.*

Definition (G., Helmick, Walsh, 2020)

1. Start from a graph $G$.
2. Construct a system of monomial equations from $G$.
3. Take the jets of order $s$ of the system.
4. Form a graph from the monomials in the jet equations.

The resulting graph $\mathcal{J}_s(G)$ is the graph of $s$-jets of $G$. 
Jets of graphs

\[ G \]
\[ x \quad y \quad xy \]

\[ \mathcal{J}_1(G) \]
\[ x_0 \quad y_0 \quad x_0y_0 + (x_0y_1 + x_1y_0)t \]
\[ x_1 \quad y_1 \quad (x_0 + x_1t)(y_0 + y_1t) \]

\[ \mathcal{J}_2(G) \]
\[ x_0 \quad y_0 \quad x_0y_0 + (x_0y_1 + x_1y_0)t + (x_0y_2 + x_1y_1x_2y_0)t^2 \]
\[ x_1 \quad y_1 \quad (x_0 + x_1t + x_2t^2)(y_0 + y_1t + y_2t^2) \]
\[ x_2 \quad y_2 \]
Jets of graphs

$\mathcal{J}_2(G)$

$G$
Jets of graphs

\[ J_2(G) \]
The **chromatic number** is the smallest number of colors needed to color a graph’s vertices so that adjacent vertices have different colors.

**Theorem (G., Helmick, Walsh, 2020)**

$\mathcal{J}_s(G)$ has the same chromatic number as $G$. 

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Diagram: 

- $G$: A line graph with vertices $w, x, y, z$. 
- $\mathcal{J}_2(G)$: A more complex graph with additional vertices and edges, showing the relationship between $G$ and its jet.
A graph is *very well covered* if all its minimal vertex covers contain exactly half of its vertices. The complete bipartite graphs $K_{n,n}$ are very well covered with 2 minimal vertex covers.

Theorem (G., Helmick, Walsh, 2020)

$J_s(K_{n,n})$ is very well covered with $s + 2$ minimal vertex covers.
Conjecture (G., Helmick, Walsh, 2020)

The jets of a very well covered graph are very well covered.

Favaron’s very well covered graph $G_1$
A graph $G$ is co-chordal if every cycle of length 4 or more in the complement of $G$ has a chord.

**Theorem (Fröberg, 1990)**

$G$ is co-chordal if and only if the minimal relations among the monomials associated to the edges of $G$ are linear.
Theorem (G., Helmick, Walsh, 2020)

*If a graph contains a path of length 3 or more, then its jets are not co-chordal.*
Theorem (G., Helmick, Walsh, 2020)

The jets of complete graphs $K_n$ and star graphs $K_{1,n}$ are co-chordal.
THANK YOU