Finite group actions on free resolutions

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Homological Methods in Commutative Algebra
AMS Spring Eastern Sectional Meeting (formerly at Tufts)
March 20, 2022

The setting

- $\Bbbk$ field
- $R = \Bbbk[x_1, \ldots, x_n]$ with a positive grading
- M finitely generated graded R-module
- $F_\bullet$ minimal free resolution of M

Proposition

If G is a group acting reasonably on R and M, then the action of G extends to $F_\bullet$.

Reasonably means: G is linearly reductive, acts $\Bbbk$-linearly on R and M, preservers degree, and distributes over multiplication.

The problem

- $\mathbb{k}$ field
- $R = \mathbb{k}[x_1, \dots, x_n]$ with a positive grading
- M finitely generated graded R-module
- $F_\bullet$ minimal free resolution of M
- $G \circlearrowright F_\bullet$

Questions

1. Can we determine the action of G on $F_\bullet$?
2. Can we describe $\mathrm{Tor}_i^R(M, \mathbb{k})_j$ as a G-representation?
3. Can we answer 1 and 2 computationally?

Motivations

A group action on a resolution may help to:

- construct the resolution;
- describe Betti numbers combinatorially;
- describe differentials.

Some big success stories rely on group actions:

- determinantal ideals (Lascoux, 1978);
- pure resolutions (Eisenbud-Fløystad-Weyman, 2011);
- Kempf-Lascoux-Weyman geometric method.

Note: the examples above use “large” groups.

A finite group example

Theorem (G., 2020)

Let I_d be the ideal generated by all squarefree monomials of degree d in $\Bbbk[x_1, \ldots, x_n]$. The number of standard Young tableaux of shape $(d, 1^i)$ with entries from 1 to n is equal to $\beta_{i,d+i}(I_d)$.

Example ($d = 2, n = 4$)

$$0 \leftarrow I_2 \leftarrow R(-2)^6 \leftarrow R(-3)^8 \leftarrow R(-4)^3 \leftarrow 0$$

$$\ytableaushort{12} \quad \ytableaushort{13} \quad \ytableaushort{23} \quad \ytableaushort{14} \quad \ytableaushort{24} \quad \ytableaushort{34}$$

$$\ytableaushort{12,3} \quad \ytableaushort{13,2} \quad \ytableaushort{12,4} \quad \ytableaushort{14,2} \quad \ytableaushort{13,4} \quad \ytableaushort{14,3} \quad \ytableaushort{23,4} \quad \ytableaushort{24,3}$$

$$\ytableaushort{12,3,4} \quad \ytableaushort{13,2,4} \quad \ytableaushort{14,2,3}$$

Interest in finite group actions on resolutions

1. Berkesch, Griffeth, Sam. Jack polynomials as fractional quantum Hall states and the Betti numbers of the $(k+1)$-equals ideal. *Comm. Math. Phys.*, 330(1):415–434, 2014.
2. G., Geramita, Wehlau. Symmetric complete intersections. *Comm. Algebra*, 46(5):2194–2204, 2018.
3. Bauer, Di Rocco, Harbourne, Huizenga, Seceleanu, Szemberg. Negative curves on symmetric blowups of the projective plane, resurgences, and Waldschmidt constants. *Int. Math. Res. Not. IMRN*, (24):7459–7514, 2019.
4. Biermann, de Alba, G., Murai, Nagel, O'Keefe, Römer, Seceleanu. Betti numbers of symmetric shifted ideals. *J. Algebra*, 560:312–342, 2020.
5. Murai. Betti tables of monomial ideals fixed by permutations of the variables. *Trans. Amer. Math. Soc.*, 373(10):7087–7107, 2020.
6. Shibata, Yanagawa. Elementary construction of minimal free resolutions of the Specht ideals of shapes $(n-2,2)$ and $(d,d,1)$, 2020, arXiv:2010.06522.
7. Raicu. Regularity of $\mathfrak{S}_n$-invariant monomial ideals. *J. Combin. Theory Ser. A*, 177:105307, 2021.
8. Murai, Raicu. An equivariant Hochster's formula for $\mathfrak{S}_n$-invariant monomial ideals. *J. Lond. Math. Soc.*, 2022.

What can we currently do?

I have an algorithm to determine the action of a semisimple Lie group on a minimal free resolution (in characteristic zero).

- G., Propagating weights of tori along free resolutions. *J. Symbolic Comput.*, 74:1–45, 2016.
- G., Free resolutions and modules with a semisimple Lie group action. *J. Softw. Algebra Geom.*, 7:17–29, 2015.

INPUT: weights of a maximal torus on ring variables and module generators.

OUTPUT: highest weights decomposition of modules in the resolution.

Question

What about finite groups?

Character Theory

- G, a finite group
- V, a finite dimensional G-representation
- $\chi_V : G \to \Bbbk$, $\chi_V(g) = \operatorname{trace}(g \colon V \to V)$, character of V
- $\forall g, h \in G$, $\chi_V(ghg^{-1}) = \chi_V(h)$

Example

$\mathfrak{S}_4 \circlearrowright V = \langle x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_3x_4 \rangle_\Bbbk$

σ	(1234)	(123)	$(12)(34)$	(12)	$\mathrm{id}_{\mathfrak{S}_4}$
$\chi_V(\sigma)$	0	0	2	2	6

Theorem

Assume $\operatorname{char}(\Bbbk) \nmid |G|$. *Two G-representations V and W are isomorphic if and only if $\chi_V = \chi_W$.*

Idea!

Example

- $R = \mathbb{C}[x_1, \dots, x_4]$
- $I = \langle x_1x_2, x_1x_3, x_2x_3, x_1x_4, x_2x_4, x_3x_4 \rangle$
- $\sigma = (12)(34)$

$$\begin{tikzcd}[column sep=huge]
R & F_1 \arrow[l, "{\begin{bmatrix} x_1x_2 & x_2x_4 & x_1x_4 & x_2x_3 & x_1x_3 & x_3x_4 \end{bmatrix}}"'] \\
R \arrow[u, "{[1]}"'] & F_1 \arrow[l, "{\begin{bmatrix} x_1x_2 & x_1x_3 & x_2x_3 & x_1x_4 & x_2x_4 & x_3x_4 \end{bmatrix}}"] \arrow[u, "{\begin{bmatrix} 1&0&0&0&0&0\\ 0&0&0&0&1&0\\ 0&0&0&1&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 0&0&0&0&0&1 \end{bmatrix}}"']
\end{tikzcd}$$

Therefore $\chi_{(F_1/\mathfrak{m}F_1)_2}(\sigma) = 2$.

Lifting actions

- $I \subseteq R$ a G-stable ideal
- $(F_\bullet, d_\bullet)$ minimal free resolution of R/I
- $\forall g \in G$, define $d_i^g \colon F_i \to F_{i-1}$ by $d_i^g = g^{-1} d_i$

Theorem (G., 2022)

For every $g \in G$,

1. *$(F_\bullet, d_\bullet^g)$ is a minimal free resolution of R/I.*
2. *If $\psi_\bullet^g \colon (F_\bullet, d_\bullet) \to (F_\bullet, d_\bullet^g)$ lifts the identity, then $\chi_{(F_i/\mathfrak{m}F_i)_j}(g)$ is equal to the trace of ψ_i^g in degree j.*

This approach is independent of the group.

Betti characters

Definition

The *Betti character* $\beta^G_{i,j}(M)$ is the character of G on $\operatorname{Tor}^R_i(M,\Bbbk)_j$ (previously denoted $\chi_{(F_i/\mathfrak{m}F_i)_j}$ if $F_\bullet$ resolves M).

1. F. Galetto. Finite group characters on free resolutions. *Journal of Symbolic Computation*, 113:29–38, 2022.
2. F. Galetto. Setting the scene for Betti characters, 2021, arXiv:2106.16062.

Macaulay2 Demo