Jets of Monomial Ideals

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Collaborators and References

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References:

- F. Galetto, E. Helmick, and M. Walsh. Jet graphs. *Involve*, 14(5):793–812, 2021.
- F. Galetto and N. Iammarino. Computing with jets. *Journal of Software for Algebra and Geometry*, 12(1):43–49, 2022.
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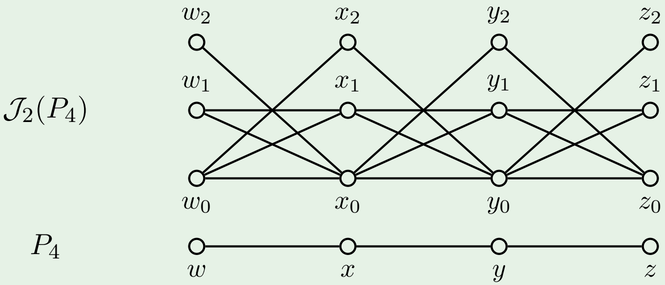
Jet graphs

Definition (-)

Let G be a simple graph and $s \in \mathbb{N}$. The *graph of s-jets of G*, denoted by $\mathcal{J}_s(G)$, has:

- vertices $x_0, \dots, x_s$ for every vertex x of G; and
- edges $\{x_i, y_j\}$ for every edge $\{x, y\}$ of G and $i, j \in \mathbb{N}$ with $i + j \leqslant s$.

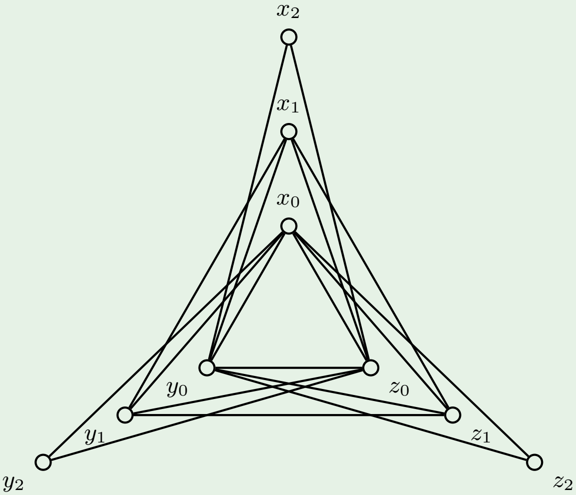
Example



Jet graphs

Example

$\mathcal{J}_2(K_3)$



Jets as differential approximations

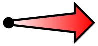
A point moves along a plane curve defined by a polynomial equation $f(x, y) = 0$.
Motion data can be expressed as Taylor polynomials in a "time" parameter t.

0-jet: point

(x_0, y_0)

1-jet: point + velocity

$(x_0 + x_1 t, y_0 + y_1 t)$



2-jet: point + velocity + acceleration

$(x_0 + x_1 t + x_2 t^2, y_0 + y_1 t + y_2 t^2)$

An s-jet is represented by $(x_0 + x_1 t + x_2 t^2 + \cdots + x_s t^s, y_0 + y_1 t + y_2 t^2 + \cdots + y_s t^s)$.
To belong on the curve, it must satisfy:

$$f\left(x_0 + x_1 t + x_2 t^2 + \cdots + x_s t^s, y_0 + y_1 t + y_2 t^2 + \cdots + y_s t^s\right) = 0$$
$$\iff f_0 + f_1 t + f_2 t^2 + \cdots + f_s t^s + \cdots = 0 \implies f_0 = f_1 = \cdots = f_s = 0,$$

where $f_0, f_1, \ldots, f_s$ are polynomials in the variables x_i, y_i obtained by expanding f and collecting powers of t.

Example

First order jets of $y^2 - x^3 = 0$ must satisfy:

$$\begin{aligned} &(y_0 + y_1 t)^2 - (x_0 + x_1 t)^3 = 0 \\ \rightsquigarrow\ &(y_0^2 - x_0^3) + (2y_0y_1 - 3x_0^2x_1)t + \cdots = 0 \\ \rightsquigarrow\ &(y_0^2 - x_0^3) = (2y_0y_1 - 3x_0^2x_1) = 0. \end{aligned}$$

The first equation is just the curve, which can be parametrized by $(x_0, y_0) = (u^2, u^3)$. When $u \neq 0$, the second equation gives:

$$2u^3y_1 - 3u^4x_1 = 0 \Longrightarrow 2y_1 - 3ux_1 = 0 \Longrightarrow y_1 = \tfrac{3}{2}ux_1.$$

Hence, the set of solutions has two components:

$$\{x_0 = y_0 = 0\} \cup \overline{\left\{\left(u^2, u^3, v, \tfrac{3}{2}uv\right)\right\}}.$$

Jet schemes

Let $I = \langle f_1, \ldots, f_r \rangle$ be an ideal in a polynomial ring over a field. Let $s \in \mathbb{N}$.

Definition

- For each variable x apply the substitution $x \longmapsto x_0 + x_1 t + x_2 t^2 + \cdots + x_s t^s$.
- After substitution, $f_i \longmapsto f_{i,0} + f_{i,1} t + f_{i,2} t^2 + \cdots + f_{i,s} t^s + \ldots$.

We call $\mathcal{J}_s(I) = \langle f_{i,j} \colon 1 \leqslant i \leqslant r, 0 \leqslant j \leqslant s \rangle$ the *ideal of s-jets* of I.

If $X = \mathbb{V}(I)$, then $\mathcal{J}_s(X) = \mathbb{V}(\mathcal{J}_s(I))$ is the scheme of s-jets of X. In particular:

- $\mathcal{J}_0(X) \cong X$;
- $\mathcal{J}_1(X)$ is the tangent bundle of X, i.e., the union of all tangent spaces of X.

Motivations and questions

The notion of jets comes from differential geometry. Its use in algebraic geometry was first suggested by John Nash as a tool to study singularities.

- If X is smooth and irreducible, then $\mathcal{J}_s(X)$ is smooth and irreducible for all $s \in \mathbb{N}$.
- If X is not smooth, $\mathcal{J}_s(X)$ may have components even when X is irreducible.

Problem

Given a variety X, describe the components of $\mathcal{J}_s(X)$.

Partial answers are available for specific varieties.

- Determinantal varieties of generic matrices (Yuen, Košir-Sethuraman, Docampo)
- Pfaffian varieties of skew-symmetric matrices (De Negri-Sbarra)
- Monomial hypersurfaces (Goward-Smith, Yuen)

Jets of monomial ideals

Jets of monomial ideals may not be monomial.

Example

If $I = \langle xy \rangle$, then $\mathcal{J}_1(I) = \langle x_0y_0, x_0y_1 + x_1y_0 \rangle$ because

$$xy \rightsquigarrow (x_0 + x_1t)(y_0 + y_1t) = x_0y_0 + (x_0y_1 + x_1y_0)t + \dots$$

Theorem (Goward, Smith, 2006)

If I is a monomial ideal, then $\sqrt{\mathcal{J}_s(I)}$ is a (squarefree) monomial ideal for every $s \in \mathbb{N}$.

If I is squarefree, the generators of $\sqrt{\mathcal{J}_s(I)}$ and I have the same degrees.

Definition (-)

Let G be a graph and $I(G)$ its edge ideal. For $s \in \mathbb{N}$, the graph defined by $\sqrt{\mathcal{J}_s(I(G))}$ is called the *graph of s-jets of G* and is denoted $\mathcal{J}_s(G)$.

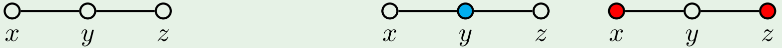
A *vertex cover* of a graph G is a set of vertices W such that every edge of G intersects W. A *minimal vertex cover* is one without vertex covers as proper subsets.

Proposition

The minimal vertex covers W of a graph G are in bijection with the minimal primes $\langle W \rangle$ of its edge ideal $I(G)$.

Example

$I(P_3) = \langle xy, yz \rangle = \langle y \rangle \cap \langle x, z \rangle$, so P_3 has minimal vertex covers $\{y\}$ and $\{x, z\}$.



Note: $\mathcal{J}_s(I(G))$ and $\sqrt{\mathcal{J}_s(I(G))}$ have the same minimal primes.

Vertex covers of jet graphs

The *cover ideal of a graph* G, denoted $I_c(G)$, is the squarefree monomial ideal generated by the products of the vertices in the minimal vertex covers of G.

Proposition

Let $I_c(G)^{(k)}$ *be the* k*-th symbolic power of* $I_c(G)$*. For every* $k \in \mathbb{N}$*,*

$$I_c(G)^{(k)} = \bigcap_{\{x,y\} \in E(G)} \langle x, y \rangle^k.$$

Example

For $I(K_3) = \langle xy, xz, yz \rangle = \langle x, y \rangle \cap \langle x, z \rangle \cap \langle y, z \rangle$, we have

$$I_c(K_3) = \langle xy, xz, yz \rangle, \qquad I_c(K_3)^{(2)} = \langle x^2y^2, x^2z^2, y^2z^2, xyz \rangle.$$

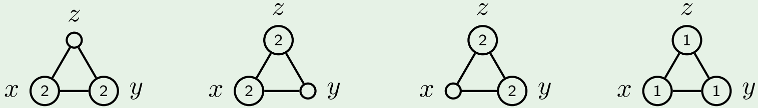
Vertex covers of jet graphs

Proposition (-)

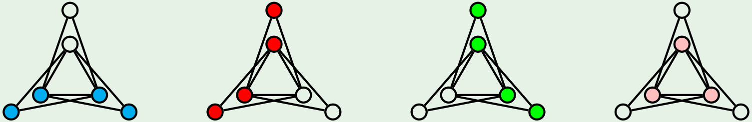
Minimal generators of $I_c(\mathcal{J}_s(G))$ are in bijection with minimal generators of $I_c(G)^{(s+1)}$.

Example

$I_c(K_3)^{(2)} = \langle x^2y^2, x^2z^2, y^2z^2, xyz \rangle$



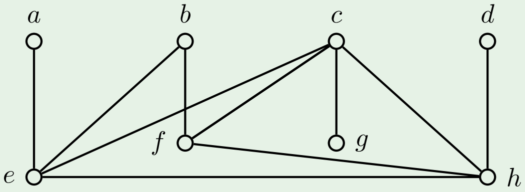
$I_c(\mathcal{J}_1(K_3)) = \langle x_0x_1y_0y_1, x_0x_1z_0z_1, y_0y_1z_0z_1, x_0y_0z_0 \rangle$



Jets of very well-covered graphs

A graph is *very well-covered* if all its minimal vertex covers contain half of the vertices.

Example (Favaron's very well-covered graph G_1)



We have $I_c(G_1) = \langle cdef, abch, bceh, cefh, efgh \rangle$.

Theorem (-; Seyed Fakhari, 2018)

The jet graphs of a very well-covered graph are very well-covered.

We use Dupont-Villarreal's generators of the symbolic Rees algebra $\mathcal{R}_s(I_c(G)) = \bigoplus_{n\in\mathbb{N}} I_c(G)^{(n)} t^n$, and properties of very well-covered graphs.

Principal jets of monomial ideals

Definition (Kosir-Sethuraman, Ghorpade-Jonov-Sethuraman, -)

For a variety $X \subseteq \mathbb{A}^n$, the *principal component* of X of order s, denoted $\mathcal{P}_s(X)$, is the closure of the s-jets of the smooth locus of X. If I is the ideal defining X, we denote $\mathcal{P}_s(I)$ the ideal defining $\mathcal{P}_s(X)$, and call it the *ideal of principal s-jets* of I.

Example

Let X be the plane curve defined by $I = \langle y^2 - x^3 \rangle$. We have

$$\mathcal{P}_1(X) = \overline{\left\{ \left(u^2, u^3, v, \tfrac{3}{2}uv \right) \right\}}.$$

By elimination, we find

$$\mathcal{P}_1(I) = \langle 3x_1y_0 - 2x_0y_1, 9x_0x_1^2 - 4y_1^2, 3x_0^2x_1 - 2y_0y_1, x_0^3 - y_0^2 \rangle.$$

Principal jets of monomial ideals

Theorem (-)

If G is a graph with minimal vertex covers $W_1, \dots, W_n$, then for every $s \in \mathbb{N}$

$$\mathcal{P}_s(I(G)) = \mathcal{J}_s(\langle W_1 \rangle) \cap \cdots \cap \mathcal{J}_s(\langle W_n \rangle).$$

Example

For $I(P_3) = \langle xy, yz \rangle = \langle y \rangle \cap \langle x, z \rangle$, we have

$$\mathcal{P}_1(I(P_3)) = \langle y_0, y_1 \rangle \cap \langle x_0, x_1, z_0, z_1 \rangle,$$
$$\mathcal{P}_2(I(P_3)) = \langle y_0, y_1, y_2 \rangle \cap \langle x_0, x_1, x_2, z_0, z_1, z_2 \rangle.$$

Corollary (-)

For every $s \in \mathbb{N}$, we have

$$\mathcal{P}_s(I(G)) = \langle x_i y_j : \{x, y\} \in E(G), 0 \leqslant i, j \leqslant s \rangle.$$

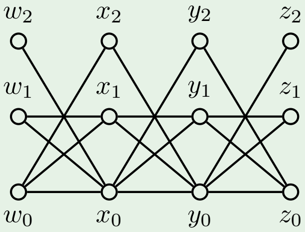
Principal jets of monomial ideals

Definition (-)

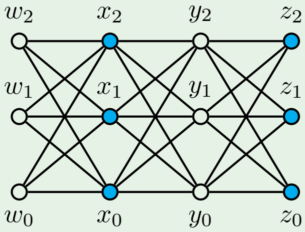
Let G be a graph and $I(G)$ its edge ideal. For $s \in \mathbb{N}$, the graph defined by $\mathcal{P}_s(I(G))$ is called the *graph of principal s-jets of G* and is denoted $\mathcal{P}_s(G)$.

Example

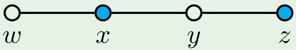
$\mathcal{J}_2(P_4)$



$\mathcal{P}_2(P_4)$



P_4

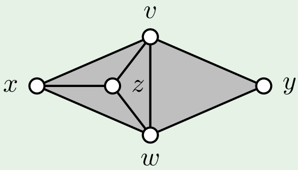


Hilbert series of principal jets of monomial ideals

Let I be the Stanley-Reisner ideal of a simplicial complex Δ. The *f-vector* of Δ is $f(\Delta) = (f_{-1}, f_0, f_1, \dots, f_i, \dots)$, where f_i is the number of i-dimensional faces of Δ.

Example

The simplicial complex Δ in the following picture



has Stanley-Reisner ideal $I_\Delta = \langle xy, yz, vwx \rangle$ and f-vector $f(\Delta) = (1, 5, 8, 4)$.

If $d = \dim(\Delta) + 1$, then the Hilbert series of the Stanley-Reisner ring $\Bbbk[\Delta]$ is given by

$$\frac{1}{(1-t)^d} \sum_{i=0}^{d} f_{i-1}(\Delta) t^i (1-t)^{d-i}.$$

Hilbert series of principal jets of monomial ideals

Theorem (-)

Let I be the Stanley-Reisner ideal of a simplicial complex Δ, and let Γ_s be the simplicial complex of $\mathcal{P}_s(I)$. For every $s \in \mathbb{N}$, there is a matrix L_s such that $f(\Gamma_s) = f(\Delta)L_s$.

Example

For $I_\Delta = \langle xy, yz, vwx \rangle$, we have $f(\Delta) = (1, 5, 8, 4)$ and

$$f(\Gamma_1) = \begin{pmatrix} 1 & 5 & 8 & 4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 12 & 6 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 10 & 37 & 64 & 56 & 24 & 4 \end{pmatrix}.$$

Corollary (-)

For every $s \in \mathbb{N}$, the multiplicity of $\mathcal{P}_s(I)$ and I is the same.

Betti numbers of principal jets of monomial ideals

Theorem (-)

Define the matrix of Betti numbers $B(\Delta) = [\beta_{j-i,j}(\Bbbk[\Delta])]_{0 \leqslant i \leqslant j}$*, and similarly for* Γ_s*. For every* $s \in \mathbb{N}$*, there is a matrix* L_s *such that* $B(\Gamma_s) = B(\Delta) L_s$*.*

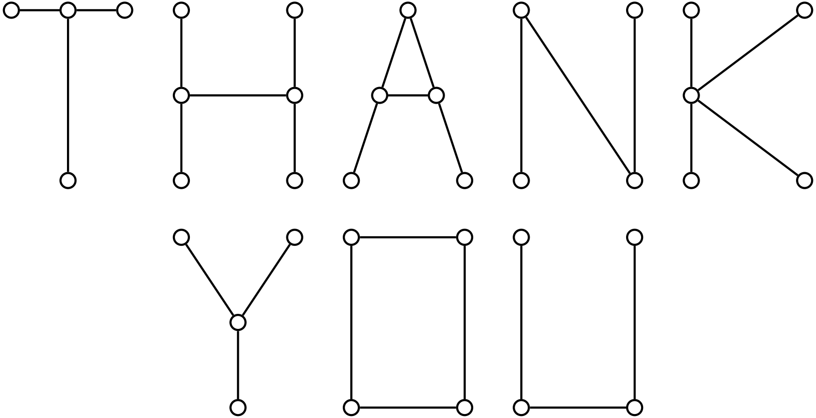
Example

For $I_\Delta = \langle xy, yz, vwx \rangle$, we have

$$B(\Gamma_1) = B(\Delta) L_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 4 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 8 & 12 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 16 & 32 & 24 & 8 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 8 & 16 & 14 & 6 & 1 & 0 & 0 \\ 0 & 0 & 0 & 8 & 28 & 38 & 25 & 8 & 1 \end{pmatrix}.$$

Corollary (-)

For every $s \in \mathbb{N}$*, the Castelnuovo-Mumford regularity of* $\mathcal{P}_s(I)$ *and* I *is the same.*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YOU

Lifting function and lifting matrix

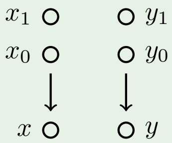
Fix a set X and $s \in \mathbb{N}$. The function $\ell_s \colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ counts the number $\ell_s(j,k)$ of ways a set U of cardinality j lifts to a set V of cardinality k along the map

$$\bigcup_{x \in X} \{x_0, x_1, \ldots, x_s\} \longrightarrow X, \qquad x_i \longmapsto x.$$

Note that $\ell_s(j,k)$ does not depend on the cardinality of X.

Example

$$\ell_1(1,1) = 2, \quad \ell_1(1,2) = 1$$
$$\ell_1(2,1) = 0, \quad \ell_1(2,2) = 4$$



Definition (-)

We call ℓ_s the *order s lifting function*, and $L_s = [\ell_s(j,k)]_{j,k \geqslant 0}$ the *order s lifting matrix*.